An Axiomatic Approach to Constructing Distances for Rank Comparison and Aggregation

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***Abstract*— We propose a new family of distance measures on rankings, derived through an axiomatic approach, that consider the nonuniform relevance of the top and bottom of ordered lists and similarities between candidates. The proposed distance functions include specialized weighted versions of the Kendall τ distance and the Cayley distance, and are suitable for comparing rankings in a number of applications, including information retrieval and rank aggregation. In addition to proposing the distance measures and providing the theoretical underpinnings for their applications, we also analyze algorithmic and computational aspects of weighted distance-based rank aggregation. We present an aggregation method based on approximating weighted distance measures by a generalized version of Spearman's footrule distance as well as a Markov chain method inspired by PageRank, where transition probabilities of the Markov chain reflect the chosen weighted distances.**

***Index Terms*— Weighted Kendall distance, positional relevance, top-vs-bottom, similarity, rank aggregation, information retrieval, statistics, collaborative filtering, PageRank.**

I. Introduction

Because of their data reduction properties, independence of scale, and ease of acquisition and representation, ordinal data structures and rankings have gained significant attention as information representation formats, with diverse applications in statistics [1]–[3], information retrieval [4], [5], social choice theory [6]–[8], coding theory [9], [10], recommender systems [11], and bioinformatics [12].

Most applications of rank processing call for a suitable notion of distance: In statistics, many variants of distance measures are used to measure correlation between rankings [1]. In information retrieval, distances are used for evaluating the accuracy of search engine results, and for comparing and aggregating them in the form of metasearch engine lists. In collaborative filtering, distances on rankings are used to measure similarity of preference lists [13], while in coding theory, they are used for assessing leakage errors and rank modulation error-correcting code construction [9], [14]. In distance-based rank aggregation, distances on rankings are central to finding aggregates or representatives of rankings, whose quality and applicability depends on the properties of the chosen distance measure [6], [15], [16].

One of the most commonly used distances on rankings is the Kendall τ distance [1], which is defined as the *smallest* number of swaps of adjacent elements that transform one ranking into the other. For example, the Kendall τ distance between the rankings $(1, 3, 4, 2)$ and $(1, 2, 3, 4)$ is two; we may first swap 2 and 4 and then 2 and 3. Besides its use in social choice theory [6] and computer science [16], the Kendall τ distance has also received significant attention in the coding theory literature due to its applications in modulation coding for flash memories [9], [14].

Despite the significant role of distance measures on rankings in ordinal data processing, the Kendall τ and other conventional distances have significant shortcomings which impair their wide-scale use in practice [5], [17]–[19]; namely, they do not take into account the varying relevance of different positions in the rankings – for example they ignore the fact that top of a ranking is typically more important than other positions – nor do they consider the similarities and dissimilarities between items in the rankings. In this work, we present new families of distance measures to resolve these issues and study their application to rank aggregation. While application-wise our focus in this paper is mainly on the rank aggregation problem, the proposed distance measures are nevertheless useful in a variety of areas such as statistics, collaborative filtering, and search engine evaluation.

The problem of rank aggregation can be succinctly described as follows: a set of "voters" or "experts" is presented with a set of candidates (objects, individuals, movies, etc.). Each voter's task is to produce a *ranking*, that is, an arrangement of the candidates in which the candidates are *ranked* from the most preferred to the least preferred. The voters' rankings are then passed to an aggregator, which outputs a single ranking, termed the aggregate ranking, to be used as a representative of all votes. Rank aggregation has applications in many fields including the social sciences, web search and Internet service studies, bioinformatics, expert opinion analysis, and economics [6], [16], [19]–[22].

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A number of rank aggregation methods have been proposed in the past, including *score-based* approaches and *distance-based* approaches. In score-based methods, the first variant of which was proposed by Borda [23], each candidate is assigned a score based on its position in each of the rankings. The candidates are then ordered based on their total score. One argument in support of Borda's method is that it ranks highly those candidates supported at least to a certain extent by almost all voters, rather than candidates who are ranked highly only by a majority of voters. In distance-based methods [6], the aggregate is defined as the ranking "closest" to the set of votes, or equivalently, at the smallest cumulative distance from the votes. Closeness of two rankings is measured via some adequately chosen distance function. Well-known distance measures for rank aggregation include the Kendall τ and Spearman's Footrule distance [24].

The choice of the distance has a significant effect on the quality and properties of the outcome of distance-based rank aggregation. This makes the problem of choosing an appropriate distance for rank aggregation applications both practically important and technically challenging. To address this issue, Kemeny [6], [25] presented a set of intuitively justifiable axioms that a distance measure must satisfy to be deemed suitable for aggregation purposes, and showed that only one distance measure satisfies these axioms – namely, the Kendall τ distance. Kemeny's set of axioms is the starting point of our construction of distances.

A. Related Work

The shortcomings of conventional distance measures have been noted by a number of authors, and various solutions to overcome the underlying issues were subsequently reported in [5], [17]–[19], [26], and [27]. In [18], Shieh presented a measure of discordance between permutations that is a generalization of the Kendall τ metric. The measure, however, is not symmetric and thus not a distance function – which introduces a number of conceptual problems. Yilmaz et al. [5] proposed a probabilistic distance measure, which in fact can be shown to be a special case of Shieh's metric [28]. A heuristic approach for addressing the top versus bottom problem was also proposed by Kumar et al. [19]. Sun et al. introduced a weighted version of Spearman's footrule in [29]. The problem of comparing the top k elements, where k is a certain positive integer, was studied by Fagin et al. [30] and in a more general context in [31]. There, the authors introduce a modification of the Kendall τ metric, which again is not a distance function. More recently, Vigna [32] proposed a weighted correlation measure on rankings with ties. In addition to being able to handle ties, which are prevalent in many applications, this correlation measure has the advantage of fast computation ($O(n \log n)$, where n is the length of the rankings). In the context of sorting and rearrangement for applications in bioinformatics, the authors of the present work studied the weighted transposition distances in terms of finding the minimum cost of sorting a permutation with cost-constrained transpositions [33]. The related problem of sorting and selection when comparisons have random costs is studied in [34] and references therein. Additionally, a distance measure taking similarities of candidates into consideration was described in [21], but with a goal opposite to ensuring diversity – the underlying distance provides heuristic guarantees that similar items are ranked close to each other in the aggregate ranking.

Our work differs from all the aforementioned contributions in two fundamental aspects. First, we rigorously derive and justify a family of distance measures based on a set of rational axioms, similar in essence to those used by Kemeny [6] to define the Kendall τ. Second, the distance measures proposed here have a level of intuitiveness and generality not matched by any previously known distance measure: the distances can be associated with adjacent and non-adjacent swaps, and can incorporate different information regarding the relevance of positions or properties of the elements to be ranked. In addition, unlike most of the aforementioned previous work, the distance measures presented here are inherently true metrics and are thus symmetric, thereby eliminating the need for choosing one ranking as ground truth or symmetrizing the distances. Furthermore, despite their generality, the proposed distances can be approximated or computed exactly in polynomial time, depending on the weights associated with the swaps. These computational performance guarantees allow for integrating the distance measures into various existing and some newly developed rank aggregation schemes [6], [16]. As a result, besides their applications in computer science and social choice theory, the developed distance measures may be used in a variety of other applications, ranging from bioinformatics to network analysis [12], [33].

B. Outline of Paper

The paper is organized as follows. We explain the motivation for our work in more detail in Section II. An overview of relevant concepts, definitions, and terminology is presented in Section III. In Section IV, we discuss Kemeny's set of axioms. More specifically, we start by showing that the set of axioms used by Kemeny in [6] to define a distance on weak orders is redundant for linear orders. We then relax and modify Kemeny's axioms in Section V to prove that there is a unique family of distances, termed weighted Kendall distances, that satisfy the modified set of axioms. Furthermore, we describe how weighted Kendall distances can address a shortcoming of conventional distances by reflecting the varying importance of different positions in rankings. While computing the weighted Kendall distances between two permutations appears to be a difficult task in general, for the important special case of monotone weights we present an efficient algorithmic solution. Furthermore, for the general model, we present a 2-approximation. We also illustrate the effect of using weighted Kendall distances as opposed to the Kendall τ in solving rank aggregation problems.

Weighted transposition distances are analyzed in detail in Section VI. These distances can take into account information about the similarity of candidates. Computing the weighted transposition distance also appears to be a difficult task. We present a 4-approximation for the general case, and a 2-approximations for specialized metric weights. We further show that for so called metric-path distances, the exact distance may be computed in polynomial time.

Section VII studies rank aggregation with weighted distances. There, we describe the performance of an algorithm for rank aggregation based on a generalization of Spearman's footrule distance and using a minimum weight matching problem (this algorithm is inspired by a procedure described in [16]) and a combination of the matching algorithm with local descent methods. Furthermore, we describe an algorithm reminiscent of PageRank [16], where the "hyperlink probabilities" are chosen according to weights.

II. Motivation

A. Top vs. Bottom

Consider the ranking π of the "World's 10 best cities to live in", according to a report composed by the Economist Intelligence Unit [35]:

$$\pi = (\text{Melbourne, Vienna, Vancouver, Toronto, Calgary, Adelaide, Sydney, Helsinki, Perth, Auckland})$$

Now consider two other rankings that both differ from π by one swap of adjacent entries:

$$\pi' = (\text{Melbourne, Vienna, Vancouver, Calgary, Toronto, Adelaide, Sydney, Helsinki, Perth, Auckland}),$$
$$\pi'' = (\text{Vienna, Melbourne, Vancouver, Toronto, Calgary, Adelaide, Sydney, Helsinki, Perth, Auckland}).$$

The astute reader probably immediately noticed that the top candidate was changed in π'', but otherwise took some time to realize where the adjacent swap appeared in π'. This is a consequence of the well-known fact that humans pay more attention to the top of the list rather than any other location in the ranking, and hence notice changes in higher positions easier.[1] Note that the Kendall τ distance between π and π' and between π and π'' is one, but it would appear reasonable to require that the distance between π and π'' be larger than that between π and π', as the corresponding swap occurred in a *more significant* (higher ranked) position in the list.

The second example corresponds to the well-studied notion of *Click-through rates* (CTRs) of webpages in search engine results pages (SERPs). The CTR is used to assess the popularity of a webpage or the success rate of an online ad. It may be roughly defined as the number of times a link is clicked on divided by the total number of times that it appeared. A recent study by Optify Inc. [36] showed that the difference between the average CTR of the first (highest-ranked) result and the average CTR of the second (runner-up) result is very large, and much larger than the corresponding difference between the average CTRs of the lower ranked items (See Figure 1). Hence, in terms of directing search engine traffic, swapping higher-ranked adjacent pairs of search results has a larger effect on the performance of Internet services than swapping lower-ranked search results.

[1]Note that one may argue that people are equally drawn to explore the highest and lowest ranked items in a list. For example, if about a hundred cities were ranked, it would be reasonable to assume that readers would be more interested in knowing the best and worst ten cities, rather than the cities occupying positions 41 to 60. These positional differences may also be addressed within the framework proposed in the paper.

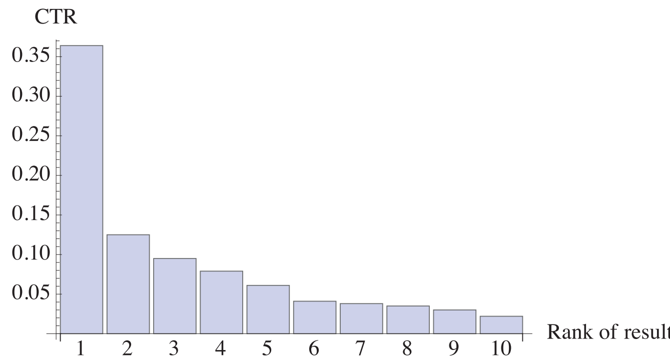


Fig. 1. Click-through rates (CTRs) of webpages appearing on the first page of Google search.

The aforementioned findings should be considered when forming an aggregate ranking of webpages. For example, in studies of CTRs, one is often faced with questions regarding traffic flow from search engines to webpages. One may think of a set of keywords, each producing a different ranking of possible webpages, with the aggregate representing the median ranking based on different sets of keywords. Based on Figure 1, if the ranking of a webpage is in the bottom half, its exact position is not as relevant as when it is ranked in the top half. Furthermore, a webpage appearing roughly half of the time at the top and roughly half of the time at the bottom will generate more incoming traffic than a webpage with persistent average ranking.

Throughout the paper, we refer to the aforementioned problem as the "top-vs-bottom" problem. To address this problem, we propose distances that penalize perturbations at the top of the list more than those at the bottom of the list by assigning different weights to swaps in different positions. These distances have another important application in practice – eliminating negative outliers. As will be shown in subsequent sections, top-vs-bottom distance measures allow candidates to be highly ranked in the aggregate even though they have a certain (small) number of highly negative ratings. The policy of eliminating outliers before rating items or individuals is a well-known one, but has not been considered in the social choice literature in the context of distance-based rank aggregation.

B. Similarity of Candidates

In many vote aggregation problems, the identities and characteristics of candidates are unknown or unimportant and all candidates are treated equally. On the other hand, many other applications require that the identities of the candidates be revealed. In the latter case, candidates are frequently partitioned in terms of some similarity criteria – for example, area of expertise, gender, working hour schedule, etc. Hence, pairs of candidates may have different degrees of similarity and swapping candidates that are similar should be penalized less than swapping candidates that are not similar.

Pertaining to the Economist Intelligence Unit ranking, it may be observed that the swap in π'' involves cities on two different continents, which may shift the general opinion about the cities' countries of origin. On the other hand, the two cities swapped in π' are both in Canada, so that the swap is not likely to change the perception of quality of living in that country. This example also points to the need for distance measures that take into account similarities and dissimilarities among candidates.

The similarity problem may be addressed through the use of a generalization of the Cayley distance between two rankings. The Cayley distance is the *smallest* number of (not necessarily adjacent) swaps required to transform one ranking into the other. For example, the Cayley distance between the permutations $(1, 2, 3, 4)$ and $(1, 4, 3, 2)$ is one, since the former can be transformed into the latter by swapping the elements 2 and 4. In contrast to the Kendall τ distance that allows for swaps of adjacent elements only, the Cayley distance allows for arbitrary swaps. Similarity of items may be captured by assigning costs or weights to swaps, and choosing the weights so that swapping dissimilar items induces a higher weight/distance compared to swapping similar items. This approach is the topic of Section VI.

III. Preliminaries

Formally, a ranking is a list of candidates arranged in order of preference, with the first candidate being the most preferred and the last candidate being the least preferred.

Consider the set of all possible rankings of a set of n candidates. Via an arbitrary, but fixed, injective mapping from the set of candidates to $\{1, 2, \ldots, n\} = [n]$, each ranking may be represented as a permutation. The mapping is often implicit and we usually equate rankings of n candidates with permutations in $\mathbb{S}_n$, where $\mathbb{S}_n$ denotes the set of permutations of $[n]$. This is equivalent to assuming that the set of candidates is the set $[n]$. For notational convenience, we use Greek lower-case letters for permutations (except the identity permutation), and explicitly write permutations as ordered sets $\sigma = (\sigma(1), \ldots, \sigma(n))$.

The identity permutation $(1, 2, \ldots, n)$ is denoted by e. For two permutations $\pi, \sigma \in \mathbb{S}_n$, the product $\mu = \pi\sigma$ is defined by the equation $\mu(i) = \pi(\sigma(i)), i = 1, 2, \cdots n$.

Definition 1: A *transposition* $\tau = (a\,b)$, for $a, b \in [n]$ and $a \neq b$, is a permutation that swaps a and b and keeps all other elements of e fixed. That is, $\tau(a) = b$, $\tau(b) = a$, and $\tau(i) = i$ for $i \notin \{a, b\}$. If $|a - b| = 1$, the transposition is referred to as an *adjacent transposition.*

Note that for $\pi \in \mathbb{S}_n$, $\pi\,(a\,b)$ is obtained from π by swapping elements in positions a and b, and $(a\,b)\,\pi$ is obtained by swapping a and b in π. For example, $(3, 1, 4, 2)(2\,3) = (3, 4, 1, 2)$ and $(2\,3)(3, 1, 4, 2) = (2, 1, 4, 3)$.

For our future analysis, we define the set

$$A(\pi, \sigma) = \big\{\tau = (\tau_1, \ldots, \tau_{|\tau|}): \sigma = \pi\tau_1 \cdots \tau_{|\tau|}, \tau_i = (a_i\, a_i + 1), i \in [|\tau|]\big\},$$

i.e., the set of all ordered sequences of adjacent transpositions that *transform* π into σ. The fact that $A(\pi, \sigma)$ is non-empty, for any $\pi, \sigma \in \mathbb{S}_n$, is obvious. Using $A(\pi, \sigma)$, the Kendall τ distance between two permutations π and σ, denoted by $K(\pi, \sigma)$, may be written as

$$K(\pi, \sigma) = \min_{\tau \in A(\pi,\sigma)} |\tau|.$$

For a ranking $\pi \in \mathbb{S}_n$ and $a, b \in [n]$, π is said to *rank a before b* or *higher than b* if $\pi^{-1}(a) < \pi^{-1}(b)$. We denote this relationship as $a <_\pi b$. Two rankings π and σ *agree* on the relative order of a pair $\{a, b\}$ of elements if both rank a before b or both rank b before a. Furthermore, the two rankings π and σ *disagree* on the relative order of a pair $\{a, b\}$ if one ranks a before b and the other ranks b before a. For example, consider $\pi = (1, 2, 3, 4)$ and $\sigma = (4, 2, 1, 3)$. We have that $4 <_\sigma 1$ and that π and σ agree on $\{2, 3\}$ but disagree on $\{1, 2\}$.

Given a distance function d over the permutations in $\mathbb{S}_n$ and a set $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$ of m votes (rankings), the distance-based aggregation problem can be stated as follows: find the ranking π^* that minimizes the cumulative distance from Σ, i.e.,

$$\pi^* = \arg\min_{\pi \in \mathbb{S}_n} \sum_{i=1}^{m} \mathsf{d}(\pi, \sigma_i). \tag{1}$$

In words, the goal is to find a ranking π that represents the median of the set of permutations Σ. The choice of the distance function d is an important aspect of distance-based rank aggregation and the focus of the paper.

IV. The Redundancy of Kemeny's Axioms

In [6], Kemeny presented a set of axioms that a distance function appropriate for rank aggregation should satisfy and proved that the only distance that satisfies the axioms is the Kendall τ. In what follows, we state the axioms and prove through a sequence of results that the postulates are redundant for linear orders.

A critical concept in Kemeny's axioms is the idea of "betweenness," defined next.

Definition 2: A ranking ω is said to be *between* two rankings π and σ, denoted by π–ω–σ, if for each pair of elements $\{a, b\}$, ω agrees with π or σ or both. The rankings $\pi_1, \pi_2, \ldots, \pi_s$ are said to be *on a line*, denoted by π_1–π_2–$\cdots$–π_s, if for every i, j, and k for which $1 \le i < j < k \le s$, we have π_i–π_j–π_k.

In Kemeny's derivations, rankings were allowed to have ties. The basis of our subsequent analysis is the same set of axioms, listed in what follows. However, our focus is on rankings without ties, or in other words, on permutations.

Axioms I

1) d is a metric.
2) d is left-invariant.
3) For any π, σ, and ω, $\mathsf{d}(\pi, \sigma) = \mathsf{d}(\pi, \omega) + \mathsf{d}(\omega, \sigma)$ if and only if ω is between π and σ.
4) The smallest positive distance is one.

Axiom I.2 states that relabeling of objects should not change the distance between permutations. In other words, $\mathsf{d}(\sigma\pi, \sigma\omega) = \mathsf{d}(\pi, \omega)$, for any $\pi, \sigma, \omega \in \mathbb{S}_n$. Axiom I.3 may be viewed through a geometric lens: the triangle inequality has to be satisfied with equality for all points that lie on a line between π and σ. Axiom I.4 is used for normalization purposes.

Kemeny's original exposition included a fifth axiom which we state for completeness: If two rankings π and σ agree except on a segment of k elements, the position of the segment does not affect the distance between the rankings. Here, a segment represents a set of objects that are ranked consecutively – i.e., a substring of the permutation. As an example, this axiom implies that

$$\begin{aligned}&\mathsf{d}((1,2,3,\underbrace{4,5,6}),(1,2,3,\underbrace{6,5,4}))\\&\quad=\mathsf{d}((1,\underbrace{4,5,6},2,3),(1,\underbrace{6,5,4},2,3))\end{aligned}$$

where the segment is underscored by braces. This axiom clearly enforces a property that *is not desirable* for metrics designed to address the top-vs-bottom issue: changing the position of the segment in two permutations does not alter their mutual distance. One may hence assume that removing this axiom (as was done in Axioms I) will lead to distance measures capable of handling the top-vs-bottom problem. But as we show, for rankings without ties, omitting this axiom does not change the outcome of Kemeny's analysis. In other words, the axiom is redundant. This is a rather surprising fact, and we conjecture that the same is true of rankings with ties.

The main result of this section is Theorem 1, which states that the unique distance satisfying Axioms I is the Kendall τ distance. In other words, the result establishes that for rankings without ties, Kemeny's fifth axiom is redundant. The theorem is proved with the help of Lemmas 1 and 2. Specifically, Lemma 1 shows that as a consequence of Axiom I.3, all adjacent transpositions have the same distance to the identity. This fact is used in Lemma 2 to prove that a distance that satisfies Axioms I, simply counts the minimum number of adjacent transpositions needed to transform one permutation into another, establishing the uniqueness statement of Theorem 1.

Lemma 1: For any d *that satisfies Axioms I, the distance between all adjacent transpositions and the identity is the same. That is, for* $i \in [n-1]$, *we have* $\mathsf{d}((i\,i+1),e) = \mathsf{d}((1\,2),e)$.

Proof: We show that $\mathsf{d}((2\,3),e) = \mathsf{d}((1\,2),e)$. Repeating the same argument used for proving this special case gives $\mathsf{d}((i\,i+1),e) = \mathsf{d}((i-1\,i),e) = \cdots = \mathsf{d}((1\,2),e)$.

To show that $\mathsf{d}((2\,3),e) = \mathsf{d}((1\,2),e)$, we evaluate $\mathsf{d}(\pi,e)$ in two ways, where we choose $\pi = (3,2,1,4,5,\ldots,n)$.

On the one hand, note that $\pi–\omega–\eta–e$, where $\omega = \pi(1\,2) = (2,3,1,4,5,\ldots,n)$ and $\eta = \omega(2\,3) = (2,1,3,4,5,\ldots,n)$. As a result,

$$\begin{aligned}\mathsf{d}(\pi,e) &= \mathsf{d}(\pi,\omega)+\mathsf{d}(\omega,\eta)+\mathsf{d}(\eta,e)\\&= \mathsf{d}(\omega^{-1}\pi,e)+\mathsf{d}(\eta^{-1}\omega,e)+\mathsf{d}(\eta,e)\\&= \mathsf{d}((1\,2),e)+\mathsf{d}((2\,3),e)+\mathsf{d}((1\,2),e)\end{aligned}\tag{2}$$

where the first equality follows from Axiom I.3, while the second is a consequence of the left-invariance property (Axiom I.2) of the distance measure.

On the other hand, note that $\pi–\alpha–\beta–e$, where $\alpha = \pi(2\,3) = (3,1,2,4,5,\ldots,n)$ and $\beta = \alpha(1\,2) = (1,3,2,4,5,\ldots,n)$. For this case,

$$\begin{aligned}\mathsf{d}(\pi,e) &= \mathsf{d}(\pi,\alpha)+\mathsf{d}(\alpha,\beta)+\mathsf{d}(\beta,e)\\&= \mathsf{d}(\alpha^{-1}\pi,e)+\mathsf{d}(\beta^{-1}\alpha,e)+\mathsf{d}(\beta,e)\\&= \mathsf{d}((2\,3),e)+\mathsf{d}((1\,2),e)+\mathsf{d}((2\,3),e).\end{aligned}\tag{3}$$

Equations (2) and (3) imply $\mathsf{d}((2\,3),e) = \mathsf{d}((1\,2),e)$. □

Lemma 2 (Uniqueness for Axioms I): For any d *satisfying Axioms I and for permutations* π *and* σ, $\mathsf{d}(\pi,\sigma)$ *equals* $K(\pi,\sigma)$, *i.e., the minimum number of adjacent transpositions required to transform* π *into* σ.

Proof: Let γ be any permutation. We first prove the lemma for the special case of $\pi = \gamma$ and $\sigma = e$. Let

$$\begin{aligned}L(\pi,\sigma) = \{\tau = (\tau_1,\ldots,\tau_{|\tau|}) \in A(\pi,\sigma):\\ \pi–\pi\tau_1–\pi\tau_1\tau_2–\cdots–\sigma\}\end{aligned}$$

be the subset of $A(\pi,\sigma)$ consisting of sequences of transpositions that transform π into σ by passing through a line. Let s be the minimum number of adjacent transpositions that transform γ into e. Furthermore, let $(\tau_1,\tau_2,\ldots,\tau_s) \in A(\gamma,e)$ and define $\gamma_i = \gamma\tau_1\cdots\tau_i, i = 0,\ldots,s$, with $\gamma_0 = \gamma$ and $\gamma_s = e$.

First, we show $\gamma_0–\gamma_1–\cdots–\gamma_s$, that is,

$$(\tau_1,\tau_2,\ldots,\tau_s) \in L(\gamma,e).\tag{4}$$

Suppose this were not the case. Then, there exist $i<j<k$ such that γ_i, γ_j, and γ_k are not on a line, and thus, there exists a pair of integer values $\{r,s\}$, $r \neq s$ for which γ_j disagrees with both γ_i and γ_k. Hence, we have two transpositions, $\tau_{i'}$ and $\tau_{j'}$, with $i < i' \le j$ and $j < j' \le k$ that swap r and s. We can in this case remove $\tau_{i'}$ and $\tau_{j'}$ from $(\tau_1,\ldots,\tau_s)$ to obtain $(\tau_1,\ldots,\tau_{i'-1},\tau_{i'+1},\ldots,\tau_{j'-1},\tau_{j'+1},\tau_s) \in A(\gamma,e)$ with length $s-2$. This contradicts the optimality of the choice of s. Hence, $(\tau_1,\tau_2,\ldots,\tau_s) \in L(\gamma,e)$. Then, by using Axiom I.3 inductively, we arrive at

$$\mathsf{d}(\gamma,e) = \sum_{i=1}^{s}\mathsf{d}(\tau_i,e).\tag{5}$$

Lemma 1 states that all adjacent transpositions have the same distance from the identity. Since transpositions τ_i, $1 \le i \le s$, in (5) are adjacent transpositions, $\mathsf{d}(\tau_i,e) = a$ for some $a > 0$ and thus $\mathsf{d}(\gamma,e) = sa$.

In (5), the minimum positive distance is obtained when $s = 1$. That is, the minimum positive distance from identity equals a and is obtained when γ is an adjacent transposition. Axiom I.4 states that the minimum positive distance is 1. By left-invariance, this axiom implies that the minimum positive distance of any permutation from the identity is 1. Hence, $a = 1$ and for any $\gamma \in \mathbb{S}_n$,

$$\mathsf{d}(\gamma,e) = \sum_{i=1}^{s}\mathsf{d}(\tau_i,e) = sa = s,$$

which completes the proof of the special case. To prove the claim for general π and σ it suffices to put $\gamma = \sigma^{-1}\pi$ and use the left-invariance property. □

For $\pi,\sigma \in \mathbb{S}_n$, let $I(\pi,\sigma) = \{\{i,j\}: i <_\pi j, j <_\sigma i\}$ be the set of pairs $\{i,j\}$ on which π and σ disagree. The number $|I(\pi,\sigma)|$ is usually referred to as the *inversion number* between the two permutations π and σ. It is well known that

$$K(\pi,\sigma) = |I(\pi,\sigma)|.\tag{6}$$

Theorem 1: The unique distance d *that satisfies Axioms I is*

$$\mathsf{d}(\pi,\sigma) = K(\pi,\sigma).$$

Proof: We only show that K satisfies Axiom I.3, as proving that K satisfies the other axioms is straightforward. Uniqueness follows from Lemma 2.

To show that K satisfies Axiom I.3, we use (6) stating that $K(\pi, \sigma) = |I(\pi, \sigma)|$. Fix $\pi, \sigma \in \mathbb{S}_n$. For any $\omega \in \mathbb{S}_n$, it is clear that

$$I(\pi, \sigma) \subseteq I(\pi, \omega) \cup I(\omega, \sigma). \tag{7}$$

Suppose first that ω is not between π and σ. Then there exists a pair $\{a, b\}$ with $a <_\pi b$ and $a <_\sigma b$, but such that $a >_\omega b$. Since $\{a, b\} \notin I(\pi, \sigma)$ and $\{a, b\} \in I(\pi, \omega) \cup I(\omega, \sigma)$, we find that

$$|I(\pi, \sigma)| < |I(\pi, \omega) \cup I(\omega, \sigma)|,$$

and thus

$$\begin{aligned} K(\pi, \sigma) &= |I(\pi, \sigma)| < |I(\pi, \omega) \cup I(\omega, \sigma)| \\ &\leq |I(\pi, \omega)| + |I(\omega, \sigma)| = K(\pi, \omega) + K(\omega, \sigma). \end{aligned}$$

Hence, if ω is not between π and σ, then

$$K(\pi, \sigma) \neq K(\pi, \omega) + K(\omega, \sigma).$$

Next, suppose ω is between π and σ. This immediately implies that $I(\pi, \omega) \subseteq I(\pi, \sigma)$ and $I(\omega, \sigma) \subseteq I(\pi, \sigma)$. These relations, along with (7), imply that

$$I(\pi, \omega) \cup I(\omega, \sigma) = I(\pi, \sigma). \tag{8}$$

We claim that $I(\pi, \omega) \cap I(\omega, \sigma) = \emptyset$. To see this to be true, observe that if $\{a, b\} \in I(\pi, \omega) \cap I(\omega, \sigma)$, then the relative rankings of a and b are the same for π and σ and so, $\{a, b\} \notin I(\pi, \sigma)$. The last statement contradicts (8) and thus

$$I(\pi, \omega) \cap I(\omega, \sigma) = \emptyset. \tag{9}$$

From (8) and (9), we may write

$$\begin{aligned} K(\pi, \sigma) &= |I(\pi, \sigma)| = |I(\pi, \omega) \cup I(\omega, \sigma)| \\ &= |I(\pi, \omega)| + |I(\omega, \sigma)| = \mathsf{d}(\pi, \omega) + \mathsf{d}(\omega, \sigma), \end{aligned}$$

and this completes the proof of the fact that K satisfies Axiom I.3. □

A distance d over $\mathbb{S}_n$ is called a *graphic distance* [37] if there exists a graph G with vertex set $\mathbb{S}_n$ such that for $\pi, \sigma \in \mathbb{S}_n$, $\mathsf{d}(\pi, \sigma)$ is equal to the length of the shortest path between π and σ in G. Note that this definition implies that the edge set of G is the set

$$\{(\alpha, \beta) : \alpha, \beta \in \mathbb{S}_n, \mathsf{d}(\alpha, \beta) = 1\}.$$

The Kendall τ distance is a graphic distance. To verify the claim, take a graph with vertices indexed by permutations, and an edge between each pair of permutations that differ by only one adjacent transposition.

In the next section, we introduce the weighted Kendall distance which may be viewed as the shortest path between permutations over a *weighted graph*, and show how this distance arises as the unique solution of a set of modified Kemeny axioms.

V. The Weighted Kendall Distance

The proof of the uniqueness of the Kendall τ distance under Axioms I, in particular, the proof of Lemma 1, reveals an important insight: the Kendall τ distance arises due to the fact that adjacent transpositions have uniform costs, which is a consequence of the betweenness property described in Axiom I.3. If one had a ranking problem in which weights of transpositions either depended on the identity of the elements involved or their positions, the uniformity assumption would have to be changed.

To change the uniformity requirement of Kemeny's axioms, we redefine the betweenness axiom, as listed in Axioms II.

Axioms II

1) d is a pseudo-metric, i.e. a generalized metric in which two distinct points may be at distance zero.
2) d is left-invariant.
3) For any π, σ disagreeing on more than one pair of elements, there exists *some* ω, distinct from π and σ and between them, such that $\mathsf{d}(\pi, \sigma) = \mathsf{d}(\pi, \omega) + \mathsf{d}(\omega, \sigma)$.

Axiom II.1 allows for the possibility of assigning distance 0 to two distinct rankings that differ in an insignificant way. Intuitively, Axiom II.3 states that there exists at least one point on some line between π and σ, for which the triangle inequality is an equality.

Definition 3: A distance d_φ is termed a *weighted Kendall distance* if there exists a nonnegative *weight function* φ over the set of adjacent transpositions such that

$$\mathsf{d}_\varphi(\pi, \sigma) = \min_{(\tau_1, \ldots, \tau_s) \in A(\pi, \sigma)} \sum_{i=1}^{s} \varphi_{\tau_i},$$

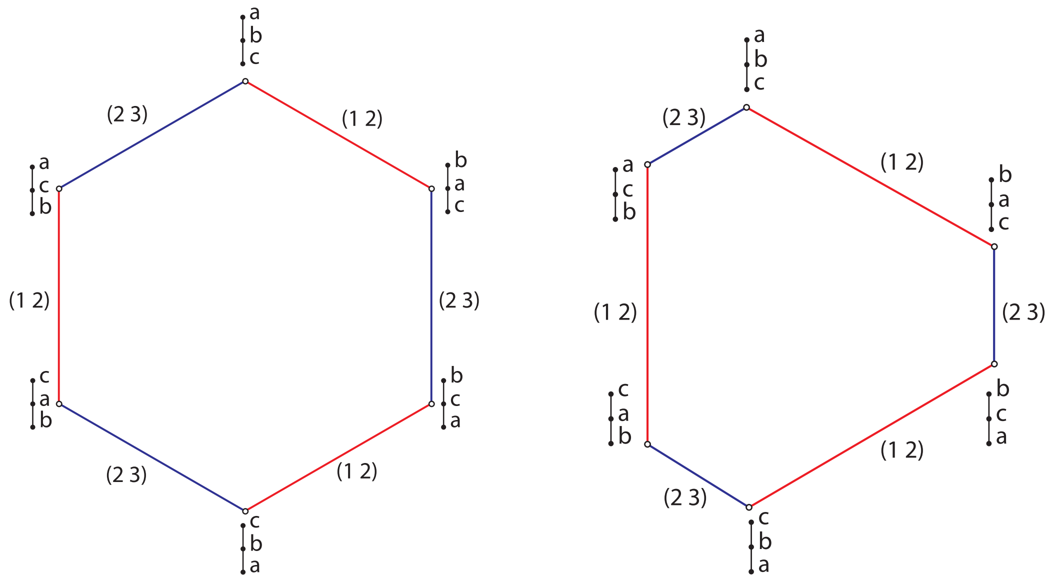
where φ_{τ_i} is the weight assigned to the adjacent transposition τ_i by φ. The weight of a transform $\tau = (\tau_1, \ldots, \tau_s)$ is denoted by $\mathrm{wt}(\tau)$ and is defined as $\mathrm{wt}(\tau) = \sum_{i=1}^{s} \varphi_{\tau_i}$. Hence, $\mathsf{d}_\varphi(\pi, \sigma)$ may be written as

$$\mathsf{d}_\varphi(\pi, \sigma) = \min_{\tau \in A(\pi, \sigma)} \mathrm{wt}(\tau).$$

Similar to the Kendall τ, the weighted Kendall distance is a graphic distance. The difference is that in the defining graph corresponding to the weighted Kendall distance, edges – which represent adjacent transpositions – are allowed to have different weights. The shortest path is obtained with respect to these weights, as illustrated in Figure 2. Note that a weighted Kendall distance is completely determined by its weight function φ.

Based on the aforementioned observations and definitions, we prove next that Axioms II ensure the existence of a unique family of distances, namely the weighted Kendall distances. This result is captured in Theorem 2. In addition, we demonstrate how an appropriately chosen weighted Kendall distance provides a simple and natural solution to the top-vs-bottom issue.

Theorem 2: A distance d satisfies Axioms II if and only if it is a weighted Kendall distance.



(a) (b)

Fig. 2. The graphs for the Kendall τ distance (a), and weighted Kendall distance (b).

Proof: We first show that for any distance d that satisfies Axioms II, and for distinct π and σ, we have

$$\mathsf{d}(\pi,\sigma)=\min_{(\tau_1,\dots,\tau_s)\in A(\pi,\sigma)}\sum_{i=1}^{s}\mathsf{d}(\tau_i,e). \tag{10}$$

The proof uses induction on $K(\pi,\sigma)$, the Kendall τ distance between π and σ. Suppose that $K(\pi,\sigma)=1$, i.e., that π and σ disagree on one pair of adjacent elements. Then, we have $\sigma=\pi(a\,a+1)$ for some $a\in[n-1]$. By left-invariance of d, it follows that $\mathsf{d}(\pi,\sigma)=\mathsf{d}((a\,a+1),e)$. We now show that the right side of (10) also equals $\mathsf{d}((a\,a+1),e)$. For every $(\tau_1,\dots,\tau_s)\in A(\pi,\sigma)$, there exists an index j, such that $\tau_j=(a\,a+1)$. This implies that

$$\min_{(\tau_1,\dots,\tau_s)\in A(\pi,\sigma)}\sum_{i=1}^{s}\mathsf{d}(\tau_i,e)\geq\mathsf{d}((a\,a+1),e). \tag{11}$$

On the other hand, since $((a\,a+1))\in A(\pi,\sigma)$,

$$\min_{(\tau_1,\dots,\tau_s)\in A(\pi,\sigma)}\sum_{i=1}^{s}\mathsf{d}(\tau_i,e)\leq\mathsf{d}((a\,a+1),e). \tag{12}$$

From (11) and (12), it follows that the right side of (10) also equals $\mathsf{d}((a\,a+1),e)$. Hence, equation (10) holds for $K(\pi,\sigma)=1$.

Now, suppose that $K(\pi,\sigma)>1$, i.e., suppose that π and σ disagree on more than one pair of elements, and that for all $\mu,\eta\in\mathbb{S}_n$ with $K(\mu,\eta)<K(\pi,\sigma)$, the lemma holds. Then, there exists an ω, distinct from π and σ and between them, such that

$$\begin{aligned}&\mathsf{d}(\pi,\sigma)=\mathsf{d}(\pi,\omega)+\mathsf{d}(\omega,\sigma),\\&K(\pi,\omega)<K(\pi,\sigma),\quad K(\omega,\sigma)<K(\pi,\sigma).\end{aligned}$$

By the induction hypothesis, there exist $(\nu_1,\dots,\nu_k)\in A(\pi,\omega)$ and $(\nu_{k+1},\dots,\nu_s)\in A(\omega,\sigma)$, for some s and k, such that $\mathsf{d}(\pi,\omega)=\sum_{i=1}^{k}\mathsf{d}(\nu_i,e)$ and $\mathsf{d}(\omega,\sigma)=\sum_{i=k+1}^{s}\mathsf{d}(\nu_i,e)$, and thus

$$\mathsf{d}(\pi,\sigma)=\sum_{i=1}^{s}\mathsf{d}(\nu_i,e)\geq\min_{(\tau_1,\dots,\tau_{s'})\in A(\pi,\sigma)}\sum_{i=1}^{s'}\mathsf{d}(\tau_i,e),$$

where the inequality follows from the fact that $(\nu_1,\dots,\nu_s)\in A(\pi,\sigma)$. To complete the proof, note that by the triangle inequality,

$$\mathsf{d}(\pi,\sigma)\leq\min_{(\tau_1,\dots,\tau_{s'})\in A(\pi,\sigma)}\sum_{i=1}^{s'}\mathsf{d}(\tau_i,e).$$

It now immediately follows that a distance d satisfying Axioms II is a weighted Kendall distance by letting $\varphi_\theta=\mathsf{d}(\theta,e)$ for every adjacent transposition θ.

The proof of the converse is omitted since it is easy to verify that a weighted Kendall distance satisfies Axioms II. $\square$

The weighted Kendall distance provides a natural solution for the top-vs-bottom issue. For instance, recall the example of ranking cities, with

π = (Melbourne, Vienna, Vancouver, Toronto, Calgary, Adelaide, Sydney, Helsinki, Perth, Auckland),

π' = (Melbourne, Vienna, Vancouver, Calgary, Toronto, Adelaide, Sydney, Helsinki, Perth, Auckland),

π'' = (Vienna, Melbourne, Vancouver, Toronto, Calgary, Adelaide, Sydney, Helsinki, Perth, Auckland),

and choose the weight function $\varphi_{(i\,i+1)}=0.9^{i-1}$ for $i=1,2,\dots,9$. Then, $\mathsf{d}_\varphi(\pi,\pi')=0.9^4=0.66<\mathsf{d}_\varphi(\pi,\pi'')=1$ as expected.

In this case, we have chosen the weight function to be exponentially decreasing in order to emphasize the importance of high rankings. In general, the choice of the weight function depends on the application at hand. For example, to evaluate aggregates of search results, one can consider the click-through rates of different ranks and the effect of errors on directing web traffic in order to identify adequate weights. For voting applications, one would need to refer to experimental studies describing how individuals perceive the values of different ranks and how much they penalize errors in different positions when evaluating the disagreement between their ranking and an alternative. For applications in bioinformatics, the weights are computed based on training data and cross-validation, as described in our companion paper [12]. An in-depth study of how one should choose the weights is out of the scope of this paper.

We next turn our attention to the computational aspects of weighted Kendall distances, addressed in Subsection V-A. There, we describe an algorithm for computing the exact weighted Kendall distances for a special, yet very important case of a weight function (summarized in Algorithm 1 and Proposition 1). In Subsection V-B, we provide a 2-approximation algorithm for the case of general weights (Proposition 2). We then proceed to illustrate the use of these distances for rank aggregation via examples provided in Subsection V-C, which show that unlike another aggregation method that uses weights to solve the top-vs-bottom issue, weighted Kendall aggregation usually satisfies the majority criterion (Proposition 3).

A. Computing the Weighted Kendall Distance for Monotonic Weight Functions

Computing the weighted Kendall distance between two permutations for an arbitrary weight function is not as straightforward a task as computing the Kendall τ distance. However, in what follows, we show that for an important class of weight functions – termed *monotonic* weight functions – the weighted Kendall distance may be computed efficiently.

Definition 4: A nonnegative weight function φ, defined on the set of adjacent transpositions, is decreasing if $i > j$ implies that $\varphi_{(i\,i+1)} \le \varphi_{(j\,j+1)}$. Increasing weight functions are defined similarly. A weight function is monotonic if it is increasing or decreasing.

Monotonic weight functions are of importance in the top-vs-bottom model as they can be used to emphasize the significance of the top of the ranking by assigning higher weights to transpositions at higher ranked positions. An example of a decreasing weight function is the exponential weight described in the previous subsection.

Suppose that $\tau = \left(\tau_1, \ldots, \tau_{|\tau|}\right)$ of length $|\tau|$ transforms π into σ. The transformation may be viewed as a sequence of moves of elements i, $i = 1, \ldots, n$, from position $\pi^{-1}(i)$ to position $\sigma^{-1}(i)$. Let the *walk* followed by element i while moved by the transform τ be denoted by $p^{i,\tau} = \left(p_1^{i,\tau}, \ldots, p_{|p^{i,\tau}|+1}^{i,\tau}\right)$, where $\left|p^{i,\tau}\right|$ is the length of the walk $p^{i,\tau}$.

For example, consider

$$\pi = (3, 2, 4, 1), \qquad \sigma = (1, 2, 3, 4),$$
$$\tau = (\tau_1, \tau_2, \tau_3, \tau_4) = ((3\,4), (2\,3), (1\,2), (2\,3))$$

and note that $\sigma = \pi\tau_1\tau_2\tau_3\tau_4$. We have

$$p^{1,\tau} = (4, 3, 2, 1), \qquad p^{2,\tau} = (2, 3, 2),$$
$$p^{3,\tau} = (1, 2, 3), \qquad p^{4,\tau} = (3, 4).$$

We first bound the lengths of the walks $p^{i,\tau}, i \in [n]$. Let $I_i(\pi, \sigma)$ be the set consisting of elements $j \in [n]$ such that π and σ disagree on the pair $\{i, j\}$. In the transform τ, all elements of $I_i(\pi, \sigma)$ must be swapped with i by some $\tau_k, k \in [|\tau|]$. Each such swap contributes length one to the total length of the walk $p^{i,\tau}$ and thus, $\left|p^{i,\tau}\right| \ge |I_i(\pi, \sigma)|$.

As before, let d_φ denote the weighted Kendall distance with weight function φ. Since for any $\tau \in A(\pi, \sigma)$,

$$\sum_{i=1}^{|\tau|} \varphi_{\tau_i} = \sum_{i=1}^{n} \frac{1}{2} \sum_{j=1}^{|p^{i,\tau}|} \varphi_{(p_j^{i,\tau}\ p_{j+1}^{i,\tau})},$$

we have

$$\mathsf{d}_\varphi(\pi, \sigma) = \min_{\tau \in A(\pi,\sigma)} \sum_{i=1}^{n} \frac{1}{2} \sum_{j=1}^{|p^{i,\tau}|} \varphi_{(p_j^{i,\tau}\ p_{j+1}^{i,\tau})}.$$

Thus,

$$\mathsf{d}_\varphi(\pi, \sigma) \ge \sum_{i=1}^{n} \frac{1}{2} \min_{p^i \in P_i(\pi,\sigma)} \sum_{j=1}^{|p^i|} \varphi_{(p_j^i\ p_{j+1}^i)}, \tag{13}$$

where for each i, $P_i(\pi, \sigma)$ denotes the set of walks of length $|I_i(\pi, \sigma)|$, starting from $\pi^{-1}(i)$ and ending in $\sigma^{-1}(i)$. For convenience, let

$$p^{i,\star}(\pi, \sigma) = \arg \min_{p^i \in P_i(\pi,\sigma)} \sum_{j=1}^{|p^i|} \varphi_{(p_j^i\ p_{j+1}^i)}$$

be the minimum weight walk from $\pi^{-1}(i)$ to $\sigma^{-1}(i)$ with length $|I_i(\pi, \sigma)|$.

If clear from the context, we write $p^{i,\star}(\pi, \sigma)$ as $p^{i,\star}$.

We show next that for decreasing weight functions, the bound given in (13) is achievable and thus the value on the right-hand-side gives the weighted Kendall distance for this class of weight functions.

Consider $\pi, \sigma \in \mathbb{S}_n$ and a decreasing weight function φ. For each i, it follows that $p^{i,\star}(\pi, \sigma)$ extends to positions with largest possible indices, i.e., $p^{i,\star} = (\pi^{-1}(i), \cdots, \ell_i - 1, \ell_i, \ell_i - 1, \ldots, \sigma^{-1}(i))$, where ℓ_i is the solution to the equation

$$\ell_i - \pi^{-1}(i) + \ell_i - \sigma^{-1}(i) = I_i(\pi, \sigma),$$

and thus $\ell_i = \left(\pi^{-1}(i) + \sigma^{-1}(i) + I_i(\pi, \sigma)\right)/2$.

We show next that there exists a transform $\tau^\star$ with $p^{i,\tau^\star} = p^{i,\star}$, and so equality in (13) can be achieved. The transform is described in Algorithm 1. The transform in question, $\tau^\star$, converts π to σ in n rounds. In Algorithm 1, the variable r takes values $\sigma(1), \sigma(2), \ldots, \sigma(n)$, in that given order. For each value of r, $\tau^\star$ moves r through a sequence of adjacent transpositions from its current position in π_t, $\pi_t^{-1}(r)$, to its position $\sigma^{-1}(r)$.

Algorithm 1 Find τ Monotone

Input: $\pi, \sigma \in \mathbb{S}_n$

Output: $\tau^\star = \arg\min_{\tau \in A(\pi,\sigma)} \mathrm{wt}(\tau)$

```
π_0 ← π
t ← 0
for r = σ(1), σ(2), ..., σ(n) do
    while π_t^{-1}(r) > σ^{-1}(r) do
        τ*_{t+1} ← (π_t^{-1}(r) − 1   π_t^{-1}(r))
        π_{t+1} ← π_t τ*_{t+1}
        t ← t + 1
    end while
end for
```

Fix $i \in [n]$. For values of r, used in Algorithm 1, such that $\sigma^{-1}(r) < \sigma^{-1}(i)$, i is swapped with r via an adjacent transposition if $\pi^{-1}(r) > \pi^{-1}(i)$. For $r = i$, i is swapped with all elements k such that $\pi^{-1}(k) < \pi^{-1}(i)$ and $\sigma^{-1}(i) < \sigma^{-1}(k)$. For r such that $\sigma^{-1}(r) > \sigma^{-1}(i)$, i is not swapped with other elements. Hence, i is swapped precisely with elements of the set $I_i(\pi,\sigma)$ and thus, $|p^{i,\tau^\star}(\pi,\sigma)| = |I_i(\pi,\sigma)|$. Furthermore, it can be seen that, for each i, $p^{i,\tau^\star}(\pi,\sigma) = (\pi^{-1}(i), \ldots, \ell_i'-1, \ell_i', \ell_i'-1, \ldots, \sigma^{-1}(i))$, for some ℓ_i'. Since $|p^{i,\tau^\star}(\pi,\sigma)| = |I_i(\pi,\sigma)|$, ℓ_i' also satisfies the equation

$$\ell_i' - \pi^{-1}(i) + \ell_i' - \sigma^{-1}(i) = I_i(\pi,\sigma),$$

implying that $\ell_i' = \ell_i$ and thus $p^{i,\tau^\star} = p^{i,\star}$. Consequently, one has the following result.

Proposition 1: For rankings $\pi, \sigma \in \mathbb{S}_n$, and a decreasing weighted Kendall weight function φ, we have

$$\mathsf{d}_\varphi(\pi,\sigma) = \sum_{i=1}^{n} \frac{1}{2}\left(\sum_{j=\pi^{-1}(i)}^{\ell_i-1} \varphi_{(j\ j+1)} + \sum_{j=\sigma^{-1}(i)}^{\ell_i-1} \varphi_{(j\ j+1)}\right)$$

where $\ell_i = \left(\pi^{-1}(i) + \sigma^{-1}(i) + I_i(\pi,\sigma)\right)/2$.

Increasing weight functions may be analyzed similarly.

Example 1: Consider the rankings $\pi = (4,3,1,2)$ and $e = (1,2,3,4)$ and a decreasing weight function φ. We have $I_i(\pi,e) = 2$ for $i = 1,2$ and $I_i(\pi,e) = 3$ for $i = 3,4$. Furthermore,

$$\ell_1 = \frac{3+1+2}{2} = 3, \quad p^{1,\star} = (3,2,1),$$
$$\ell_2 = \frac{4+2+2}{2} = 4, \quad p^{2,\star} = (4,3,2),$$
$$\ell_3 = \frac{2+3+3}{2} = 4, \quad p^{3,\star} = (2,3,4,3),$$
$$\ell_4 = \frac{1+4+3}{2} = 4, \quad p^{4,\star} = (1,2,3,4).$$

Hence, the minimum weight transformation generated by the algorithm is

$$\tau^\star = \left(\underbrace{(3\,2),(2\,1)}_{1}, \underbrace{(4\,3),(3\,2)}_{2}, \underbrace{(4\,3)}_{3}\right),$$

where the numbers under the braces denote the value r corresponding to the indicated transpositions. The distance between π and e equals

$$\mathsf{d}_\varphi(\pi,e) = \varphi_{(1\,2)} + 2\varphi_{(2\,3)} + 2\varphi_{(3\,4)}.$$

Example 2: The bound given in (13) is not tight for general weight functions. Consider $\pi = (4,2,3,1)$, $\sigma = (1,2,3,4)$, and a weight function φ with $\varphi_{(1\,2)} = 2, \varphi_{(2\,3)} = 1$, and $\varphi_{(3\,4)} = 2$. We have

$$p^{1,\star} = (4,3,2,1), \qquad p^{2,\star} = (2,3,2),$$
$$p^{3,\star} = (3,2,3), \qquad p^{4,\star} = (1,2,3,4).$$

Suppose that a transform τ exists such that $p^{i,\star} = p^{i,\tau}, i = 1,2,3,4$. From $p^{i,\star}$, it follows that in τ, transpositions $(1\,2)$ and $(3\,4)$ each appear once and $(2\,3)$ appears twice. It can be shown, by considering all possible re-orderings of $\{(1\,2),(1\,2),(2\,3),(2\,3),(2\,3)\}$ or by an application of [33, Lemma 5] that τ does not transform π into σ. Hence, for this example, the lower bound (13) is not achievable.

The results of this subsection imply that for decreasing weight functions, which capture the importance of the top entries in rankings, computing the weighted Kendall distance has time complexity $O(n^2)$. On the other hand, Knight [38] described a method for computing the Kendall τ metric in time $O(n \log n)$. More recently, Chan and Pătraşcu showed [39] that this task can in fact be performed in time $O(n\sqrt{\log n})$.[2] While, there is a gap between the performance of the algorithm given here and those for computing the Kendall τ metric, in many applications, distance computation efficiency should not impede the use of these weighted Kendall distances.

1) Weight Functions With Two Identical Non-Zero Weights: In addition to monotone weights, we describe another example of a weighted Kendall distance for which a closed form solution exists. For a pair of integers $a, b, 1 \le a < b < n$, define the weight function as:

$$\varphi_{(i\ \ i+1)} = \begin{cases} 1, & i \in \{a,b\}, \\ 0, & \text{else.} \end{cases} \tag{14}$$

Such weight functions may be used in aggregation problems where one penalizes moving a link from one page (say, top-ten page) to another page (say, ten-to-twenty page). In other words, one only penalizes moving an item from a "high-ranked" set of positions to "average-rank" or "low-rank" positions. An expression for computing the weighted Kendall distance for this case is given in the Appendix (Section VII-B).

B. Approximating the Weighted Kendall Distance for General Weight Functions

In what follows, we present a polynomial-time 2-approximation algorithm for computing the general form of weighted Kendall distances, as well as two algorithms for computing this distance exactly. While exact computations may require super-exponential time complexity, for a small number of candidates – say, less than 10 – the computation can still be performed in reasonable time. A small number

[2]We thank Sebastiano Vigna for pointing out the references regarding the time complexity of computing the Kendall τ to us.

of candidates and a large number of voters are frequently encountered in social choice applications, but less frequently in computer science.

In order to approximate the weighted Kendall distance, $\mathsf{d}_\varphi(\pi,\sigma)$, we use the function $D_\varphi(\pi,\sigma)$, defined as

$$D_\varphi(\pi,\sigma) = \sum_{i=1}^{n} w(\pi^{-1}(i):\sigma^{-1}(i)), \tag{15}$$

where

$$w(k:l) = \begin{cases} \sum_{h=k}^{l-1} \varphi_{(h\,h+1)}, & \text{if } k<l,\\ \sum_{h=l}^{k-1} \varphi_{(h\,h+1)}, & \text{if } k>l,\\ 0, & \text{if } k=l,\end{cases}$$

denotes the sum of the weights of adjacent transpositions $(k\,k+1),(k+1\,k+2),\ldots,(l-1\,l)$, if $k<l$, the sum of the weights of adjacent transpositions $(l\,l+1)$, $(l+1\,l+2),\ldots,(k-1\,k)$, if $l<k$, and 0, if $k=l$.

The following proposition states lower and upper bounds for d_φ in terms of D_φ. The proposition is useful in practice, since D_φ can be computed in time $O(n^2)$, and provides the desired 2-approximation.

Proposition 2: For a weighted Kendall weight function φ and for permutations π and σ,

$$\frac{1}{2}D_\varphi(\pi,\sigma) \le \mathsf{d}_\varphi(\pi,\sigma) \le D_\varphi(\pi,\sigma).$$

We omit the proof of the proposition, since it follows from a more general result stated in the next section, and only remark that the lower-bound presented in Proposition 2 is weaker than the lower-bound given by (13).

Next, we discuss computing the *exact* weighted Kendall distance via algorithms for finding minimum weight paths in graphs. As already pointed out, the Kendall τ and the weighted Kendall distance are graphic distances. In the latter case, we define a graph G with the vertex set equal to $\mathbb{S}_n$ and an edge of weight $\varphi_{(i\ i+1)}, i\in[n-1]$, between each pair of vertices π and σ for which there exists an i such that $\pi=\sigma(i\ i+1)$. The numbers of vertices and edges of G are $|V|=n!$ and $|E|=n!(n-1)/2$, respectively. Dijkstra's algorithm with Fibonacci heaps [40] for finding the minimum weight path in a graph provides the distances of all $\pi\in\mathbb{S}_n$ to the identity in time $O(|E|+|V|\log|V|)=O(n!\,n\log n)$.

One can actually show that the complexity of the algorithm for finding the distance between $\pi\in\mathbb{S}_n$ and the identity equals $O(n(K(\pi,e))!)$, which is significantly smaller than $\Omega(n!)$ for permutations at small Kendall τ distance. The minimum weight path algorithm is based on the following observation. For π in $\mathbb{S}_n$, there exists a transform $\tau=(\tau_1,\tau_2,\ldots,\tau_m)$ of minimum weight that transforms π into e, such that $m=K(\pi,e)$. In other words, each transposition of τ eliminates one inversion when transforming π into e. Hence, $\pi\tau_1$ has one less inversion than π. As a result,

$$\mathsf{d}_\varphi(\pi,e) = \min_{i:\pi(i)>\pi(i+1)} \left(\varphi_{(i\ i+1)} + \mathsf{d}_\varphi(\pi(i\ i+1),e)\right). \tag{16}$$

Note that the minimum is taken over all positions i for which i and $i+1$ form an inversion, i.e., for which $\pi(i)>\pi(i+1)$. Suppose that computing the weighted Kendall distance between the identity and a permutation π, with $K(\pi,e)=d$, can be performed in time T_d. From (16), we have

$$T_d = a\,n + d\,T_{d-1}, \quad \text{for } d\ge 2,$$

and $T_1 = a\,n$, for some constant a. By letting $U_d = T_d/(a\,n\,d!)$, we obtain $U_d = U_{d-1}+\frac{1}{d!}, d\ge 2$, and $U_1=1$. Hence, $U_d=\sum_{i=1}^{d}\frac{1}{i!}$. It can then be shown that $d!\,U_d = \lfloor d!(e-1)\rfloor$, and thus $T_d = a\,n\lfloor d!\,(e-1)\rfloor = O(nd!)$.

The expression (16) can also be used to find the distances of all $\pi\in\mathbb{S}_n$ from the identity by first finding the distances of permutations $\pi\in\mathbb{S}_n$ with $K(\pi,e)=1$, then finding the distances of permutations $\pi\in\mathbb{S}_n$ with $K(\pi,e)=2$, and so on.[3] Unfortunately, the average Kendall τ distance between a randomly chosen permutation and the identity is $\binom{n}{2}/2$ (see the derivation of this known and a related novel result regarding the weighted Kendall distance in the Appendix), which limits the applicability of this algorithm to uniformly and at random chosen votes.

C. Aggregation With Weighted Kendall Distances: Examples

In order to explain the potential of the weighted Kendall distance in addressing the top-vs-bottom aggregation issue, in what follows, we present a number of examples that illustrate how the choice of the weight function influences the final form of the aggregate. We focus on *decreasing weight functions* and compare our results to those obtained using the classical Kendall τ distance.

Throughout the remainder of the paper, we refer to a solution of the aggregation problem using the Kendall τ as a *Kemeny aggregate*. All the aggregation results are obtained via exhaustive search since the examples are small and only used for illustrative purposes. Aggregation is, in general, a hard problem and we postpone the analysis of the complexity of computing aggregate rankings, and aggregate approximation algorithms, until Section VII.

Example 3: Consider the set of rankings listed in Σ, where each row represents a ranking (vote),

$$\Sigma = \begin{pmatrix} 4 & 1 & 2 & 5 & 3\\ \hline 4 & 2 & 1 & 3 & 5\\ \hline 1 & 4 & 5 & 2 & 3\\ \hline 2 & 3 & 1 & 5 & 4\\ \hline 5 & 3 & 1 & 2 & 4 \end{pmatrix}.$$

The Kemeny optimal solution for this set of rankings is $(1,4,2,5,3)$. Note that despite the fact that candidate 4 was ranked twice at the top of the list – more than any other candidate – it is ranked only second in the aggregate. This may be attributed to the fact that 4 was ranked last by two voters.

Consider next the weight function $\varphi^{(2/3)}$ with $\varphi^{(2/3)}_{(i\,i+1)} = (2/3)^{i-1}, i\in[4]$. The optimum aggregate ranking for this weight equals $(4,1,2,5,3)$. The optimum aggregate based on $\varphi^{(2/3)}$ puts 4 before 1, similar to what a plurality vote

[3]Note that such an algorithm requires that the set of permutations at a given Kendall τ distance from the identity be known and easy to create/list.

would do.[4] The reason behind this swap is that $\varphi^{(2/3)}$ emphasizes strong showings of a candidate and downplays its weak showings, since weak showings have a smaller effect on the distance as the weight function is decreasing. In other words, higher ranks are more important than lower ranks when determining the position of a candidate.

Example 4: Consider the set of rankings listed in Σ,

$$\Sigma = \begin{pmatrix} 1 & 4 & 2 & 3 \\ \hline 1 & 4 & 3 & 2 \\ \hline 2 & 3 & 1 & 4 \\ \hline 4 & 2 & 3 & 1 \\ \hline 3 & 2 & 4 & 1 \end{pmatrix}.$$

The Kemeny optimal solution is $(4, 2, 3, 1)$. Note that although the majority of voters prefer 1 to 4, 1 is ranked last and 4 is ranked first. More precisely, we observe that according to the pairwise majority test, 1 beats 4 but loses to 2 and 3. On the other hand, 4 is preferred to both 2 and 3 but, as mentioned before, loses to 1. Problems like this do not arise due to a weakness of Kemeny's approach, but due to the inherent "rational intractability" of rank aggregation. As stated by Arrow [41], for *any* "reasonable" rank aggregation method, there exists a set of votes such that the aggregate ranking prefers one candidate to another while the majority of voters prefer the later to the former.

Let us now focus on a weighted Kendall distance with weight function $\varphi_{(i\,i+1)} = (2/3)^{i-1}, i = 1, 2, 3$. The optimal aggregate ranking for this distance equals $(1, 4, 2, 3)$. Again, we see a candidate with both strong showings and weak showings, candidate 1, beat a candidate with a rather average performance, candidate 4. Note that in this solution as well, there exist candidates for which the opinion of the majority is ignored: 1 is placed before 2 and 3, while according to the pairwise majority opinion it loses to both.

Example 5: Consider the set of rankings listed in Σ,

$$\Sigma = \begin{pmatrix} 5 & 4 & 1 & 3 & 2 \\ \hline 1 & 5 & 4 & 2 & 3 \\ \hline 4 & 3 & 5 & 1 & 2 \\ \hline 1 & 3 & 4 & 5 & 2 \\ \hline 4 & 2 & 5 & 3 & 1 \\ \hline 1 & 2 & 5 & 3 & 4 \\ \hline 2 & 4 & 3 & 5 & 1 \end{pmatrix}.$$

With the weight function $\varphi_{(i\,i+1)} = (2/3)^{i-1}, i \in [4]$, the aggregate equals $(4, 1, 5, 2, 3)$. The winner is 4, while the plurality rule winner is 1 as it appears three times on the top. Next, we increase the rate of decay of the weight function and let $\varphi_{(i\,i+1)} = (1/3)^{i-1}, i \in [4]$. In this case, the solution equals $(1, 4, 2, 5, 3)$, and the winner is candidate 1, the same as the plurality rule winner. This result is a consequence of the fact that the plurality winner is the aggregate based on the weighted Kendall distance with weight function $\varphi^{(p)}$,

$$\varphi^{(p)}_{(i\,i+1)} = \begin{cases} 1, & i = 1, \\ 0, & \text{else.} \end{cases}$$

The Kemeny aggregate is $(4, 5, 1, 2, 3)$.

[4]In *plurality voting*, the candidate with the most first-place rankings is declared the winner.

A shortcoming of distance-based rank aggregation is that sometimes the solution is not unique, and that the possible solutions may differ widely. The following example describes one such scenario.

Example 6: Suppose that the votes are given by Σ, where

$$\Sigma = \begin{pmatrix} 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 3 & 2 & 1 \\ \hline 2 & 1 & 3 \end{pmatrix}.$$

Here, the permutations $(1, 2, 3), (2, 1, 3)$ are the Kemeny optimal solutions, with cumulative distance 4 from Σ. When the Kemeny optimal solution is not unique, it may be possible to obtain a unique solution by using a non-uniform weight function. In this example, it can be shown that for any non-uniform weight function φ with $\varphi_{(1\,2)} > \varphi_{(2\,3)}$, the solution is unique, namely, $(1, 2, 3)$.

A similar situation occurs if the last vote is changed to $(2, 3, 1)$. In that case, the permutations $(1, 2, 3)$, $(2, 1, 3)$, and $(2, 3, 1)$ are the Kemeny optimal solutions with cumulative distance 5 from Σ. Again, for any non-uniform weight function φ with $\varphi_{(1\,2)} > \varphi_{(2\,3)}$ the solution is unique and equal to $(1, 2, 3)$.

To summarize, these examples illustrate how a proper choice for the weighted Kendall distance insures that top ranks are emphasized and how one may over-rule a moderate number of low rankings using a specialized distance formula. One may argue that certain generalizations of Borda's method, involving non-uniform gaps between ranking scores, may achieve similar goals. This is not the case, as will be illustrated in what follows.

One major difference between generalized Borda and weighted Kendall distances is in the already mentioned *majority criterion* [42], which states that the candidate ranked first by the majority of voters has to be ranked first in the aggregate.[5] Borda's aggregate with an arbitrary score assignments does not have this property, while aggregates obtained via weighted Kendall distances with decreasing weights not identically equal to zero have this property.

We first show that the Borda method with a fixed, but otherwise arbitrary set of scores may not satisfy the majority criterion. We prove this claim for $n = 3$. A similar argument can be used to establish this claim for $n > 3$.

Suppose, for simplicity, that the number m of voters is odd and that, for each vote, a score s_i is assigned to the candidate with rank i, $i = 1, 2, 3$. Here, we assume that $s_1 > s_2 > s_3 \geq 0$. Suppose also that $(m+1)/2$ of the votes equal (a, b, c) and that $(m-1)/2$ of the votes equal (b, c, a). Let the total Borda scores for candidates a and b be denoted by S and S', respectively. We have

$$S = \frac{m+1}{2} s_1 + \frac{m-1}{2} s_3,$$
$$S' = \frac{m+1}{2} s_2 + \frac{m-1}{2} s_1,$$

[5]Note that a candidate ranked first by the majority is a Condorcet candidate. It is desirable that an aggregation rule satisfy the majority criterion and indeed most do, including the Condorcet method, the plurality rule, the single transferable vote method, and the Coombs method.

and thus $S - S' = s_1 - m\left(\frac{s_2 - s_3}{2}\right) - \frac{s_2 + s_3}{2}$. If $m > \frac{2s_1 - (s_2+s_3)}{s_2 - s_3}$, then $S - S' < 0$ and Borda's method ranks b higher than a. As a result, candidates a, ranked highest by more than half of the voters, is not ranked first according to Borda's rule. This is not the case with weighted Kendall distances, as shown in the sequel.

Proposition 3: An aggregate ranking obtained using the weighted Kendall distance with a decreasing weight function not identically equal to zero satisfies the majority criterion.

Proof: Suppose that the weight function is φ, and let $w_i = \varphi_{(i\ i+1)}$. Since w is decreasing and not identically equal to zero, we have $w_1 > 0$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$ denote the set of votes and let a_1 be the candidate that is ranked first by a majority of voters. Partition the set of votes into two sets, C and D, where C is the set of votes that rank a_1 first and D is the set of votes that do not. Furthermore, denote the aggregate ranking by π.

Suppose that a_1 is not ranked first in π and that π is of the form

$$(a_2, \ldots, a_i, a_1, a_{i+1}, \ldots, a_n),$$

for some $i \geq 2$. Let $\pi' = (a_1, a_2, \ldots, a_n)$. We show that

$$\sum_{j=1}^{m} \mathsf{d}_\varphi\left(\pi, \sigma_j\right) > \sum_{j=1}^{m} \mathsf{d}_\varphi\left(\pi', \sigma_j\right)$$

which contradicts the optimality of π, implying that a_1 must be ranked first in π.

For $\sigma \in C$, we have

$$\mathsf{d}_\varphi(\pi, \sigma) = \mathsf{d}_\varphi(\pi, \pi') + \mathsf{d}_\varphi(\pi', \sigma). \tag{17}$$

To see the validity of this claim, note that if π is to be transformed to σ via Algorithm 1, it is first transformed to π' by moving a_1 to the first position.

For $\sigma \in D$, we have

$$\mathsf{d}_\varphi(\pi', \sigma) \leq \mathsf{d}_\varphi(\pi', \pi) + \mathsf{d}_\varphi(\pi, \sigma), \tag{18}$$

which follows from the triangle inequality.

To complete the proof, we write

$$\begin{aligned}
\sum_{j=1}^{m} \mathsf{d}_\varphi(\pi, \sigma_j) &= \sum_{\sigma \in C} \mathsf{d}_\varphi(\pi, \sigma) + \sum_{\sigma \in D} \mathsf{d}_\varphi(\pi, \sigma) \\
&\geq \sum_{\sigma \in C} \mathsf{d}_\varphi(\pi', \sigma) + |C| \mathsf{d}_\varphi(\pi, \pi') \\
&\quad + \sum_{\sigma \in C} \mathsf{d}_\varphi(\pi', \sigma) - |D| \mathsf{d}_\varphi(\pi, \pi') \\
&= \sum_{j=1}^{m} \mathsf{d}_\varphi(\pi', \sigma) + (|C| - |D|)\, \mathsf{d}_\varphi(\pi, \pi') \\
&> \sum_{j=1}^{m} \mathsf{d}_\varphi(\pi', \sigma)
\end{aligned}$$

where the first inequality follows from (17) and (18), and the second inequality follows from the facts that $|C| > |D|$ and that $\mathsf{d}_\varphi(\pi, \pi') \geq w_1 > 0$. □

VI. Weighted Transposition Distance

The definition of the Kendall τ distance and the weighted Kendall distance is based on transforming one permutation into another using *adjacent* transpositions. If, instead, all transpositions are allowed – including non-adjacent transpositions – a more general distance measure, termed *weighted transposition distance* is obtained. This distance measure, as will be demonstrated, represents a generalization of the weighted Kendall distance suitable for addressing similarity issues among candidates.

Definition 5: Consider a function φ that assigns to each transposition θ, a non-negative weight φ_θ. The weight of a sequence of transpositions is defined as the sum of the weights of its transpositions. That is, the weight of the sequence $\tau = (\tau_1, \ldots, \tau_{|\tau|})$ of transpositions equals

$$\mathrm{wt}(\tau) = \sum_{i=1}^{|\tau|} \varphi_{\tau_i}.$$

For simplicity, we also denote the weighted transposition distance between two permutations $\pi, \sigma \in \mathbb{S}_n$, with weight function φ, by d_φ. This distance equals the minimum weight of a sequence $\tau = (\tau_1, \ldots, \tau_{|\tau|})$ of transpositions such that $\sigma = \pi \tau_1 \cdots \tau_{|\tau|}$. As before, we refer to such a sequence of transpositions as a transform converting π to σ and let $A_T(\pi, \sigma)$ denote the set of transforms that convert π to σ.

With this notation at hand, the weighted transposition distance between π and σ, with respect to φ, may be written as

$$\mathsf{d}_\varphi(\pi, \sigma) = \min_{\tau \in A_T(\pi, \sigma)} \mathrm{wt}(\tau).$$

The Kendall τ distance and the weighted Kendall distance may be viewed as special cases of the weighted transposition distance: to obtain the Kendall τ distance, let

$$\varphi_\theta = \begin{cases} 1, & \theta = (i\ i+1), i = 1, 2, \ldots, n-1 \\ \infty, & \text{else,} \end{cases}$$

and to obtain the weighted Kendall distance, let

$$\varphi_\theta = \begin{cases} w_i, & \theta = (i\ i+1), i = 1, 2, \ldots, n-1 \\ \infty, & \text{else,} \end{cases}$$

for a non-negative weight function w. It is worth pointing out that the weighted transposition distance is not based on the axiomatic approach described in the previous section.

When applied to the inverse of rankings, the weighted transposition distance can be successfully used to model similarities of objects in rankings. In such a setting, permutations that differ by a transposition of two similar items are at a smaller distance than permutations that differ by a transposition of two dissimilar items, as demonstrated in the next subsection.

A. Weighted Transposition Distance as Similarity Distance

We illustrate the concept of distance measures taking into account similarities via the following example, already mentioned in the Motivation section. Suppose that four cities:

Melbourne, Sydney, Helsinki, and Vienna are ranked based on certain criteria as

$$\pi = (\text{Helsinki, Sydney, Vienna, Melbourne}),$$

and according to another set of criteria as

$$\sigma = (\text{Melbourne, Vienna, Helsinki, Sydney}).$$

The similarity distance between π and σ is defined as follows. We assign weights to swapping cities in the rankings, e.g., suppose that the weight of swapping cities in the same country is 1, in the same continent 2, and 3 otherwise. The *similarity distance* between π and σ is the minimum weight of a sequence of swaps that converts π to σ, where the weights are determined by the similarity of the items being swapped. By inspection, one can see that the similarity distance between π and σ equals 6. One of the sequences of swaps of weight 6 is as follows: first swap Helsinki and Sydney with weight 3, then swap Melbourne and Sydney with weight 1, and finally swap Vienna and Helsinki with weight 2.

To express the similarity distance formally, we write the rankings as permutations, representing Melbourne by 1, Sydney by 2, Vienna by 3, and Helsinki by 4. This is equivalent to assuming that the identity ranking is

$$e = (\text{Melbourne, Sydney, Vienna, Helsinki}).$$

We then have $\pi = (4, 2, 3, 1)$ and $\sigma = (1, 4, 2, 3)$.

It is straightforward to see that every statements about swapping elements i and j may be converted to statements made about swapping elements at positions i and j by using the inverse of the ranking/permutation. Therefore, we may cast the similarity distance as the weighted transposition distance. This approach has the benefit of being consistent with the weighted Kendall distance. The similarity distance between π and σ is equal to the weighted transposition distance between π^{-1} and σ^{-1} with the weight function

$$\varphi_{(1\,2)} = 1, \quad \varphi_{(1\,3)} = 3, \quad \varphi_{(1\,4)} = 3,$$
$$\varphi_{(2\,3)} = 3, \quad \varphi_{(2\,4)} = 3, \quad \varphi_{(3\,4)} = 2.$$

It should be clear from the context that the indices in the weight function refer to the candidates, and not positions.

Example 7: Consider the votes listed in Σ,

$$\Sigma = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 3 & 2 & 1 & 4 \\ \hline 4 & 1 & 3 & 2 \end{array}\right).$$

Suppose that even numbers and odd numbers represent different types of candidates in a way that the following weight function is appropriate

$$\varphi_{(i\,j)} = \begin{cases} 1, & \text{if } i, j \text{ are both odd or both even,} \\ 2, & \text{else.} \end{cases}$$

Note that the votes are "diverse" in the sense that they alternate between odd and even numbers. On the other hand, the Kemeny aggregate is $(1, 3, 2, 4)$, which puts all odd numbers ahead of all even numbers. Aggregation using the similarity distance yields $(1, 2, 3, 4)$, a solution which may be considered "diverse" since the even and odd numbers alternate in the solution. The reason behind this result is that the Kemeny optimal solution is oblivious to the identity of the candidates and their (dis)similarities, while aggregation based on similarity distances takes such information into account.

Example 8: Consider the votes listed in Σ,

$$\Sigma = \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 3 & 6 & 5 & 2 & 1 & 4 \\ \hline 3 & 6 & 5 & 2 & 1 & 4 \\ \hline 5 & 4 & 1 & 6 & 3 & 2. \end{array}\right).$$

Suppose that the weight function is the same as the one used in the previous example. In this case, neither the Kemeny aggregates nor the similarity distance aggregates are unique. More precisely, Kendall τ gives four solutions:

$$\left(\begin{array}{cccccc} 3 & 5 & 1 & 6 & 2 & 4 \\ \hline 3 & 5 & 1 & 2 & 4 & 6 \\ \hline 1 & 3 & 5 & 2 & 4 & 6. \\ \hline 1 & 3 & 5 & 6 & 2 & 4. \end{array}\right),$$

while there exist nine optimal aggregates under the weighted transposition distance of the previous example, of total distance 10:

$$\left(\begin{array}{cccccc} 5 & 6 & 3 & 4 & 1 & 2 \\ \hline 5 & 4 & 3 & 2 & 1 & 6 \\ \hline 5 & 2 & 3 & 6 & 1 & 4 \\ \hline 3 & 4 & 1 & 2 & 5 & 6 \\ \hline 3 & 6 & 1 & 4 & 5 & 2 \\ \hline 3 & 2 & 1 & 6 & 5 & 4 \\ \hline 1 & 4 & 5 & 2 & 3 & 6 \\ \hline 1 & 2 & 5 & 6 & 3 & 4 \\ \hline 1 & 6 & 5 & 4 & 3 & 2. \end{array}\right).$$

Note that *none of* the Kemeny optimal aggregates have good diversity properties: the top half of the rankings consists exclusively of odd numbers. This is because this method cannot take into account the fact that all voters picked an even number for the second position. On the other hand, the optimal similarity distance rankings *all contain* exactly one even element among the top-three candidates. Such diversity properties are hard to prove theoretically.

It is important to note that the use of similarity distance for rank aggregation, while incorporating similarity information, has the drawback of ignoring the positions of the items being swapped. Perhaps the ideal distance measure based on pairwise swaps depends on both positions and identities of candidates and not one or the other. Computing such distances however may be challenging and requires more study.

In Subsection VI-B, we study how to compute the weighted transposition distance and present a 4-approximation for this distance (see Theorem 3). We also improve this approximation result for two special cases of weight functions, as described in Propositions 4 and 5.

B. Computing the Weighted Transposition Distance

In this subsection, we describe how to compute or approximate the weighted transposition distance d_φ, given the weight function φ. Certain aspects of the computation of the weighted transposition distances are studied in more detail in our companion paper [33] in the context of sorting and rearrangement.

We find the following definitions useful in our subsequent derivations. For a given weight function φ, we let $\mathcal{K}_\varphi$ denote a complete undirected weighted graph with vertex set $[n]$, where the weight of each edge (i, j) equals the weight of the transposition $(i\ j)$, $\varphi_{(i\ j)}$. For a subgraph H of $\mathcal{K}_\varphi$, with edge set E_H, we define the weight of H as

$$\mathrm{wt}(H) = \sum_{(i,j)\in E_H} \varphi_{(i\ j)},$$

that is, the sum of the weights of edges of H. For $\pi, \sigma \in \mathbb{S}_n$, we define $D_\varphi(\pi, \sigma)$ as[6]

$$D_\varphi(\pi, \sigma) = \sum_{i=1}^{n} \mathrm{wt}\left(p^*_\varphi(\pi^{-1}(i), \sigma^{-1}(i))\right),$$

where $p^*_\varphi(a, b)$ denotes the minimum weight path from a to b in $\mathcal{K}_\varphi$.

It is easy to verify that D_φ is a pseudo-metric and that it is left-invariant,

$$D_\varphi(\eta\pi, \eta\sigma) = D_\varphi(\pi, \sigma), \qquad \pi, \sigma, \eta \in \mathbb{S}_n.$$

A weight function φ is a *metric weight function* if it satisfies the triangle inequality in the sense that

$$\varphi_{(a\,b)} \le \varphi_{(a\,c)} + \varphi_{(b\,c)}, \qquad a, b, c \in [n]. \tag{19}$$

Lemma 3: For a weight function φ and for $\pi, \sigma \in \mathbb{S}_n$, we have $\mathsf{d}_\varphi(\pi, \sigma) \le 2D_\varphi(\pi, \sigma)$. If φ is a metric weight function, the bound may be improved to $\mathsf{d}_\varphi(\pi, \sigma) \le D_\varphi(\pi, \sigma)$.

Due to its length, the proof of Lemma 6 is presented in the appendix. The next lemma provides a lower bound for d_φ in terms of D_φ.

Lemma 4: For $\pi, \sigma \in \mathbb{S}_n$, we have $\mathsf{d}_\varphi(\pi, \sigma) \ge \frac{1}{2} D_\varphi(\pi, \sigma)$.

Proof: Since d_φ and D_φ are both left-invariant, it suffices to show that $\mathsf{d}_\varphi(\pi, e) \ge \frac{1}{2} D_\varphi(\pi, e)$. Let $(\tau_1, \ldots, \tau_l)$, with $\tau_j = (a_j\, b_j)$, be a minimum weight transform of π into e, so that $\mathsf{d}_\varphi(\pi, e) = \sum_{i=1}^{l} \varphi_{(a_j\, b_j)}$. Furthermore, define $\pi_j = \pi\tau_1 \cdots \tau_j$, $0 \le j \le l$. Then,

$$D_\varphi\left(\pi_{j-1}, e\right) - D_\varphi\left(\pi_j, e\right) \le 2\mathrm{wt}\left(p^*_\varphi\left(a_j, b_j\right)\right) \le 2\varphi_{(a_j\, b_j)},$$

where the first inequality follows from considering the maximum possible decrease of the value of D_φ induced by the transposition $(a_j\, b_j)$, while the second inequality follows from the definition of p^*_φ. By summing up the terms in the preceding formula over $0 \le j \le l$, and thus obtaining a telescoping inequality of the form $D_\varphi(\pi, e) \le 2\sum_{i=1}^{l} \varphi_{(a_j\, b_j)} = 2\mathsf{d}_\varphi(\pi, e)$, we arrive at the desired result. □

From the previous two lemmas, we have the following theorem.

Theorem 3: For $\pi, \sigma \in \mathbb{S}_n$ and an arbitrary non-negative weight function φ, we have

$$\frac{1}{2} D_\varphi(\pi, \sigma) \le \mathsf{d}_\varphi(\pi, \sigma) \le 2D_\varphi(\pi, \sigma).$$

In addition, if φ is a metric weight function, then

$$\frac{1}{2} D_\varphi(\pi, \sigma) \le \mathsf{d}_\varphi(\pi, \sigma) \le D_\varphi(\pi, \sigma).$$

[6]Note that this definition is consistent with the definition of a specialization of this function, given in (15).

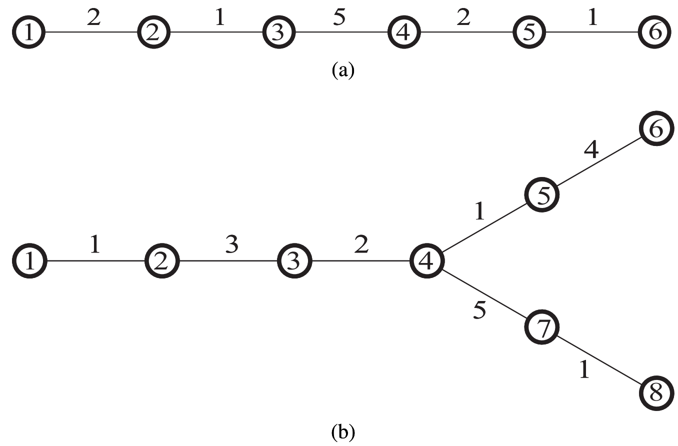


Fig. 3. A defining path (a), which may correspond to a metric-path weight function or an extended-path weight function, and a defining tree (b), which may correspond to a metric-tree weight function or an extended-tree weight function.

1) Computing the Distance for Metric-Tree Weights: For special classes of the weight function φ, described below, the bounds in Theorem 3 may be improved further. We start with the following definitions.

Definition 6: A weight function φ is a *metric-tree weight function* if there exists a weighted tree Θ over the vertex set $[n]$ such that for distinct $a, b \in [n]$, $\varphi_{(a\,b)}$ is the sum of the weights of the edges on the unique path from a to b in Θ. If Θ is a path, i.e., if Θ is a linear graph, then φ is called a *metric-path weight function*.

Furthermore, a weight function φ' is an *extended-tree weight function* if there exists a weighted tree Θ over the vertex set $[n]$ such that for distinct $a, b \in [n]$, $\varphi'_{(a\,b)}$ equals the the weight of the edge (a, b) whenever a and b are adjacent, and $\varphi'_{(a\,b)} = \infty$ otherwise. If Θ is a path, then φ' is called an *extended-path* weight function.

Note that the Kendall weight function, defined in the previous section, is an extended path weight function.

The tree or path corresponding to a weight function in the preceding definitions is termed the *defining tree or path* of the weight function. An example is given in Figure 3, where the numbers indexing the edges denote their weights.

For a metric-tree weight function φ with defining tree Θ, and for $a, b \in [n]$, the weight of the path $p^*_\varphi(a, b)$ equals the weight of the *unique* path from a to b in Θ. This weight, in turn, equals $\varphi_{(a\,b)}$. As a result, for metric-tree weights, $p^*_\varphi(a, b)$ equals the weight of the path from a to b in Θ.

Furthermore, from Lemma 4, we have $\mathsf{d}_\varphi((a\ b), e) \ge \frac{1}{2} D_\varphi((a\ b), e) = \mathrm{wt}(p^*_\varphi(a, b)) = \varphi_{(a\,b)}$. Since we also have $\mathsf{d}_\varphi((a\ b), e) \le \varphi_{(a\,b)}$, it follows that

$$\mathsf{d}_\varphi((a\ b), e) = \varphi_{(a\,b)}. \tag{20}$$

The next proposition shows that the *exact distance* for metric-path weight functions can be computed in polynomial time.

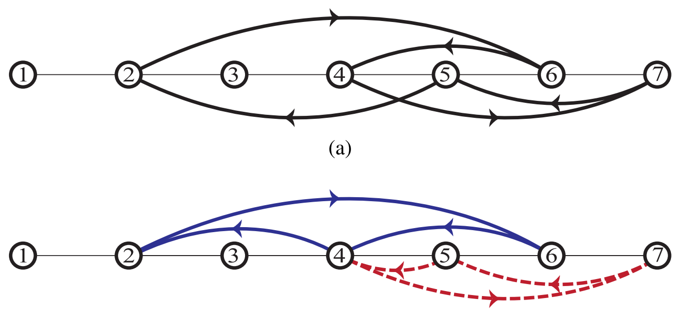


Fig. 4. The cycle (2 6 4 7 5) in Figure (a) is decomposed into two cycles, (2 6 4) and (4 7 5), depicted in Figure (b). Note that (2 6 4 7 5) = (2 6 4)(4 7 5).

Proposition 4: For a metric-path weight function φ and for $\pi, \sigma \in \mathbb{S}_n$, we have $\mathsf{d}_\varphi(\pi,\sigma) = \frac{1}{2} D_\varphi(\pi,\sigma)$.

Proof: From Lemma 4, we have that $\mathsf{d}_\varphi(\pi,\sigma) \geq \frac{1}{2} D_\varphi(\pi,\sigma)$. It remains to show that $\mathsf{d}_\varphi(\pi,\sigma) \leq \frac{1}{2} D_\varphi(\pi,\sigma)$. Since d_φ and D_φ are both left-invariant, it suffices to prove that $\mathsf{d}_\varphi(\pi, e) \leq \frac{1}{2} D_\varphi(\pi, e)$.

Let $\{c_1, c_2, \ldots, c_k\}$ be the cycle decomposition of π. Similar to the proof of Lemma 6 (see Appendix), it suffices to show that

$$\mathsf{d}_\varphi(c, e) \leq \frac{1}{2} D_\varphi(c, e) \tag{21}$$

for any cycle $c = (a_1\, a_2 \cdots a_{|c|})$.

The proof is by induction. For $|c| = 2$, (21) holds since, from (20), we have

$$\mathsf{d}_\varphi((a_1\, a_2), e) = \varphi_{(a\, b)} = \mathrm{wt}\left(p^*_\varphi(a_1, a_2)\right) = \frac{1}{2} D_\varphi((a_1\, a_2), e).$$

Assume that (21) holds for $2 \leq |c| < l$. We show that it also holds for $|c| = l$. We use Figure 4 for illustrative purposes. In all figures in this section, undirected edges describe the defining tree, while directed edges describe the cycle at hand.

Without loss of generality, assume that the defining path of φ, Θ, equals $(1, 2, \ldots, n)$. Furthermore, assume that $a_1 = \min\{i : i \in c\}$; if this were not the case, we could rewrite c by cyclically shifting its elements. Let $a_t = \min\{i : i \in c, i \neq a_1\}$ be the "closest" element of c to a_1 (other than a_1 itself). For example, in Figure 4, one has $c = (2\,6\,4\,7\,5)$, $a_1 = 2$ and $a_t = 4$.

We have

$$\begin{aligned} c &= (a_1\, a_2 \cdots a_t \cdots a_l) \\ &= (a_1\ a_2\ \cdots\ a_t)(a_t\ a_{t+1}\ \cdots\ a_l) \end{aligned}$$

and thus

$$\begin{aligned} \mathsf{d}_\varphi(c, e) &\leq \mathsf{d}_\varphi((a_1\ a_2\ \cdots\ a_t), e) + \mathsf{d}_\varphi((a_t\ a_{t+1}\ \cdots\ a_l), e) \\ &\leq \frac{1}{2} \sum_{i=1}^{t-1} \mathrm{wt}\left(p^*_\varphi(a_i, a_{i+1})\right) + \mathrm{wt}\left(p^*_\varphi(a_t, a_1)\right) \\ &\quad + \frac{1}{2} \sum_{i=t}^{l-1} \mathrm{wt}\left(p^*_\varphi(a_i, a_{i+1})\right) + \mathrm{wt}\left(p^*_\varphi(a_l, a_t)\right) \\ &= \frac{1}{2} \sum_{i=1}^{l} \mathrm{wt}\left(p^*_\varphi(a_i, c(a_i))\right) = \frac{1}{2} D_\varphi(c, e). \end{aligned}$$

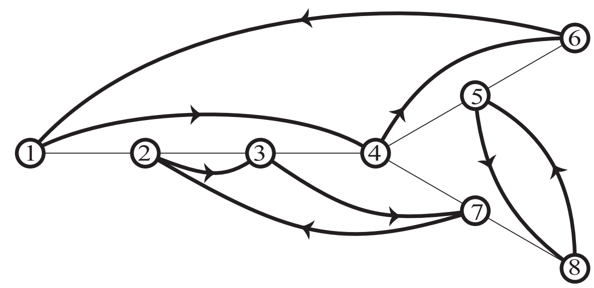


Fig. 5. If each of the cycles of a permutation lie on a path, the method of Proposition 4 can be used to find the weighted transposition distance.

where the second inequality follows from the induction hypothesis, while the first equality follows from the fact that $\mathrm{wt}\left(p^*_\varphi(a_t, a_1)\right) + \mathrm{wt}\left(p^*_\varphi(a_l, a_t)\right) = \mathrm{wt}\left(p^*_\varphi(a_l, a_1)\right)$. □

The approach described in the proof of Proposition 4 can also be applied to the problem of finding the weighted transposition distance when the weight function is a metric-tree weight function and *each of the cycles of the permutation* consist of elements that lie on some path in the defining tree. An example of such a permutation and such a weight function is shown in Figure 5. Note that in this example, a cycle consisting of elements 3, 5, 7 would not correspond to a path.

In such a case, for each cycle c of π we can use the path in the defining tree that contains the elements of c to show that

$$\mathsf{d}_\varphi(c, e) = \frac{1}{2} D_\varphi(c, e). \tag{22}$$

For example the cycle (1 4 6) lies on the path (1, 2, 3, 4, 5, 6) and the cycle (5 8) lies on the path (5, 4, 7, 8). Since (22) holds for each cycle c of π, we have

$$\mathsf{d}_\varphi(\pi, e) = \frac{1}{2} D_\varphi(\pi, e).$$

A similar scenario in which essentially the same argument as that of the proof of Proposition 4 can be used is as follows: the defining tree has one vertex with degree three and no vertices with degree larger than three (i.e., a tree with a Y shape), and for each cycle of π, there are two branches of the tree that do not contain two consecutive elements of c. It can then be shown that each such cycle can be decomposed into cycles that lie on paths in the defining tree, reducing the problem to the previously described one. An example is shown in Figure 6.

One may argue that the results of Proposition 4 and its extension to metric-trees have limited application, as both the defining tree and the permutations/rankings used in the computation must be of special forms. One way to satisfy these conditions is to require that a ranking π be such that there are *no edges* in the cycle graph of π between two given branches of a Y-shaped tree Θ. We show next that under certain conditions the probability of such permutations goes to one as $n \to \infty$.

Let the set of vertices in the ith branch of a Y shaped defining tree Θ, $i = 1, 2, 3$, be denoted by B_i and let b_i denote the number of vertices in B_i. Clearly, $b_1 + b_2 + b_3 + 1 = n$.

Assume, without loss of generality, that the numbering of the branches is such that $b_1 \geq b_2 \geq b_3$. As an illustration,

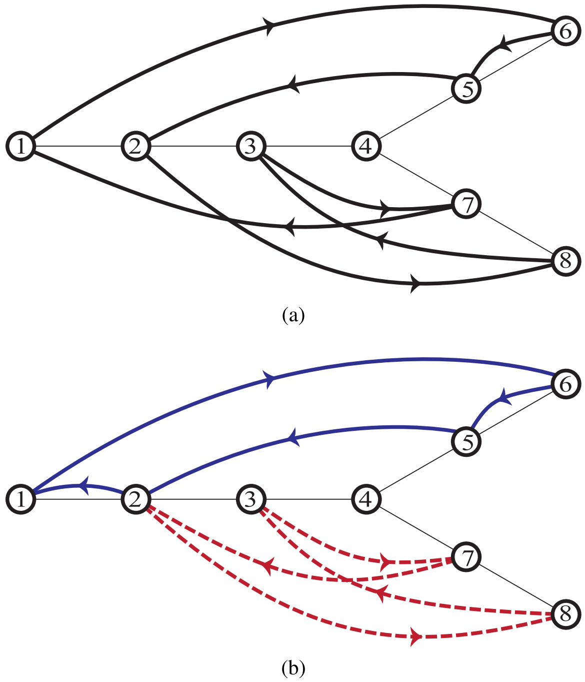


Fig. 6. The cycle (1 6 5 2 8 3 7) in Figure (a) is decomposed into two cycles, (1 6 5 2) and (2 8 3 7), as shown in Figure (b). Note that (1 6 5 2 8 3 7) = (1 6 5 2)(2 8 3 7).

in Figure 6 we have

$$\begin{aligned} B_1 &= \{1, 2, 3\}, \quad b_1 = 3,\\ B_2 &= \{5, 6\}, \quad b_2 = 2,\\ B_3 &= \{7, 8\}, \quad b_3 = 2. \end{aligned}$$

Let P_n denote the number of permutations π whose cycle decomposition does not contain an edge between B_2 and B_3. This quantity is greater than or equal to the number of permutations π such that $\pi(j) \notin B_2 \cup B_3$ for $j \in B_2 \cup B_3$. The number of permutation with the latter property equals $\binom{b_1+1}{b_2+b_3}(b_2+b_3)!(b_1+1)!$. Hence,

$$P_n \geq \frac{((b_1+1)!)^2}{(b_1+1-b_2-b_3)!},$$

and thus

$$\frac{P_n}{n!} \geq \frac{\prod_{j=n+1-2b_2-2b_3}^{n-b_2-b_3} j}{\prod_{j=n+1-b_2-b_3}^{n} j}.$$

In particular, if $b_2 = b_3 = 1$, we have

$$\frac{P_n}{n!} \geq \frac{(n-3)(n-2)}{(n-1)n} = 1 - \frac{4}{n} + O(n^{-2})$$

and more generally, if $b_2 + b_3 = o(n)$, then

$$\frac{P_n}{n!} \geq \frac{(n+o(n))^{b_2+b_3}}{(n+o(n))^{b_2+b_3}} \sim 1,$$

or equivalently, $P_n \sim n!$.

Hence, if $b_2 + b_3 = o(n)$, the distance $\mathsf{d}_\varphi(\pi, e)$ of a randomly chosen permutation π from the identity equals $D_\varphi(\pi, e)/2$ with probability approaching 1 as $n \to \infty$.

It is worth noting that for metric-tree weight functions, the equality of Proposition 4 is not, in general, satisfied.

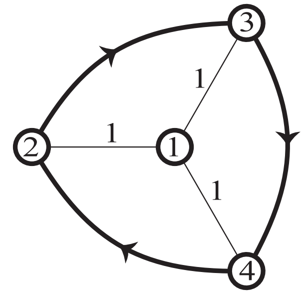


Fig. 7. For the above metric-tree weight function and $\pi = (2\,3\,4)$, the equality of Proposition 4 does not hold.

To prove this claim, consider the metric-tree weight function φ in Figure 7, where, for $a, b \in [4]$, $a < b$,

$$\varphi_{(a\,b)} = \begin{cases} 1, & \text{if } a = 1,\\ \infty, & \text{if } a \neq 1. \end{cases}$$

It can be shown that for the permutation $\pi = (2\,3\,4)$, $\mathsf{d}_\varphi(\pi, e) = 4$, while $\frac{1}{2}D_\varphi(\pi, e) = 3$.

The following proposition provides a 2-approximation for transposition distances based on extended-path weight functions. As the weighted Kendall distance is a special case of the weighted transposition distance with extended-path weight functions, the proposition also implies Proposition 2.

Proposition 5: For an extended-path weight function φ and for $\pi, \sigma \in \mathbb{S}_n$,

$$\frac{1}{2}D_\varphi(\pi, \sigma) \leq \mathsf{d}_\varphi(\pi, \sigma) \leq D_\varphi(\pi, \sigma).$$

Proof: The lower bound follows from Lemma 4. To prove the upper bound, consider a *metric*-path weight function φ', with the same defining path Θ as φ, such that

$$\varphi'_{(a\ b)} = 2\varphi_{(a\ b)}$$

for any pair a, b adjacent in Θ. From Lemma 6, it follows that for distinct $c, d \in [n]$,

$$\mathsf{d}_\varphi((c\ d), e) \leq 2\mathrm{wt}(p^*_\varphi(c, d)) = \mathrm{wt}(p^*_{\varphi'}(c, d)) = \mathsf{d}_{\varphi'}((c\ d), e).$$

Hence,

$$\mathsf{d}_\varphi(\pi, \sigma) \leq \mathsf{d}_{\varphi'}(\pi, \sigma) = \frac{1}{2}D_{\varphi'}(\pi, \sigma) = D_\varphi(\pi, \sigma),$$

which proves the claimed result. □

VII. Aggregation Algorithms

Despite the importance of the rank aggregation problem in many areas of information retrieval, only a handful of results regarding the complexity of the problem are known. Among them, the most important result is the fact that finding a Kemeny optimal solution is NP-hard [16]. Since the Kendall τ distance is a special case of the weighted Kendall distance, finding the aggregate ranking for the latter is also NP-hard. In particular, exhaustive search approaches – akin to the one we used in the previous sections – are not computationally feasible for large problems.

We present next algorithms for aggregating rankings using weighted distances. We provide an algorithm based on bipartite matching applicable to weighted transposition distances that gives a 2-approximation for aggregation under weighted Kendall distances (Proposition 6). The output of this algorithm is then used as the starting point of a local search algorithm. We also present a PageRank-like algorithm for aggregation with weighted Kendall distances.

For the first algorithmic approach, assuming that π^* is the solution to (1), the ranking σ_i closest to π^* provides a 2-approximation for the aggregate ranking. This easily follows from the fact that the Kendall τ distance satisfies the triangle inequality. As a result, one only has to evaluate the pairwise distances of the votes Σ in order to identify a 2-approximation aggregate for the problem. Assuming the weighted Kendall distance can be computed efficiently (for example, if the weight function is monotonic), the same is true of the weighted Kendall distance as it is also a metric and thus satisfies the triangle inequality.

A second method for obtaining a 2-approximation is an extension of a bipartite matching algorithm. For any distance function that may be written as

$$\mathsf{d}(\pi, \sigma) = \sum_{k=1}^{n} f(\pi^{-1}(k), \sigma^{-1}(k)), \qquad (23)$$

where f denotes an arbitrary non-negative function, one can find an *exact solution* to (1) as described in the next section. The matching algorithm approach for classical Kendall τ aggregation was first proposed in [16].

A. Vote Aggregation Using Matching Algorithms

Consider a complete weighted bipartite graph $\mathcal{G} = (X, Y)$, with $X = \{1, 2, \dots, n\}$ corresponding to the n ranks to be filled in, and $Y = \{1, 2, \dots, n\}$ corresponding to the elements of $[n]$, i.e., the candidates. Let (i, j) denote an edge between $i \in X$ and $j \in Y$. We say that a perfect bipartite matching P corresponds to a permutation π whenever $(i, j) \in P$ if and only if $\pi(i) = j$. If the weight of (i, j) equals

$$\sum_{l=1}^{m} f(i, \sigma_l^{-1}(j)),$$

i.e., the weight incurred by $\pi(i) = j$, the minimum weight perfect matching corresponds to a solution of (1). The distance of (23) is a generalized version of Spearman's footrule since Spearman's footrule [24] can be obtained by choosing $f(x, y) = |x - y|$. Below, we explain how to use the matching approach for aggregation based on a general weighted Kendall distance. More details about this approach may be found in our companion conference paper [43].

Recall that for a weighted Kendall weight function φ,

$$D_\varphi(\pi, \sigma) = \sum_{i=1}^{n} w(\pi^{-1}(i) : \sigma^{-1}(i)),$$

where

$$w(k : l) = \begin{cases} \sum_{h=k}^{l-1} \varphi_{(h\, h+1)}, & \text{if } k < l, \\ \sum_{h=l}^{k-1} \varphi_{(h\, h+1)}, & \text{if } k > l, \\ 0, & \text{if } k = l. \end{cases}$$

Note that D_φ is a distance measure of the form of (23), and thus a solution to problem (1) for $\mathsf{d} = D_\varphi$ can be found exactly in polynomial time.

Suppose that the set of votes is given by $\Sigma = \{\sigma_1, \dots, \sigma_m\}$.

Proposition 6: Let $\pi' = \arg\min_\pi \sum_{l=1}^{m} D_\varphi(\pi, \sigma_i)$ *and* $\pi^* = \arg\min_\pi \sum_{l=1}^{m} \mathsf{d}_\varphi(\pi, \sigma_i)$. *The permutation* π' *is a 2-approximation to the optimal rank aggregate* π^* *if* φ *corresponds to a weighted Kendall distance.*

Proof: From Proposition 2, for a weighted Kendall weight function φ and for permutations π and σ,

$$\frac{1}{2} D_\varphi(\pi, \sigma) \le \mathsf{d}_\varphi(\pi, \sigma) \le D_\varphi(\pi, \sigma).$$

Thus we have

$$\sum_{l=1}^{m} \mathsf{d}_\varphi(\pi', \sigma_i) \le \sum_{l=1}^{m} D_\varphi(\pi', \sigma_i).$$

and

$$\frac{1}{2} \sum_{l=1}^{m} D_\varphi(\pi^*, \sigma_i) \le \sum_{l=1}^{m} \mathsf{d}_\varphi(\pi^*, \sigma_i)$$

From the optimality of π' with respect to D, we find

$$\sum_{l=1}^{m} D_\varphi(\pi', \sigma_i) \le \sum_{l=1}^{m} D_\varphi(\pi^*, \sigma_i).$$

Hence

$$\sum_{l=1}^{m} \mathsf{d}_\varphi(\pi', \sigma_i) \le 2 \sum_{l=1}^{m} \mathsf{d}_\varphi(\pi^*, \sigma_i).$$

□

In fact, the preceding proposition applies to the larger class of weighted transposition distances with extended-path weights. It can similarly be shown that for a weighted transposition distance with general weights (resp. metric weights), π' is a 4-approximation (resp. a 2-approximation). Finally, for a weighted transposition distance with metric-path weights, π' represents the exact solution.

A simple approach for improving the performance of the matching based algorithm for the weighted Kendall distances is to couple it with a local descent method. Assume that the estimate of the aggregate at step ℓ equals $\pi^{(\ell)}$. Let $\mathbb{A}_n$ be the set of adjacent transpositions in $\mathbb{S}_n$. Then

$$\pi^{(\ell+1)} = \pi^{(\ell)} \arg\min_{\theta \in \mathbb{A}_n} \sum_{i=1}^{m} \mathsf{d}(\pi^{(\ell)}\theta, \sigma_i).$$

The search terminates when the cumulative distance of the aggregate from the set of votes Σ cannot be decreased further. We choose the starting point $\pi^{(0)}$ to be the ranking π' of Proposition 6 obtained by the minimum weight bipartite matching algorithm. This method will henceforth be referred to as Bipartite Matching with Local Search (BMLS).

An important question at this point is how does the approximate nature of the BMLS aggregation process change the aggregate, especially with respect to the top-vs-bottom or similarity property? This question is hard, and we currently have no mathematical results pertaining to this problem. Instead, we describe a number of simulation results that may guide future analysis of this issue.

In order to see the effect of the BMLS on vote aggregation, we revisit Examples 3-6. In *all except for one case* the solution provided by BMLS is the same as the exact solution, both for the Kendall τ and weighted Kendall distances.

The exception is Example 4. In this case, for the weight function $\varphi_{(i\,i+1)} = (2/3)^{i-1}, i \in [3]$, the exact solution equals $(1, 4, 2, 3)$ but the solution obtained via BMLS equals $(4, 2, 3, 1)$. Note that these two solutions differ significantly in terms of their placement of candidate 1, ranked first in the exact ranking and last in the approximate ranking. The distances between the two solutions, $\mathsf{d}_\varphi((1, 4, 2, 3), (4, 2, 3, 1))$, equals 2.11 and is rather large. Nevertheless, the cumulative distances to the votes are very close in value:

$$\sum_i \mathsf{d}_\varphi((1, 4, 2, 3), \sigma_i) = 9,$$
$$\sum_i \mathsf{d}_\varphi((4, 2, 3, 1), \sigma_i) = 9.11.$$

Hence, as with any other distance based approach, the approximation result may sometimes diverge significantly from the optimum solution while the closeness of the approximate solution to the set of votes is nearly the same as that of the optimum solution.

B. Vote Aggregation Using PageRank

An algorithm for data fusion based on the PageRank and HITS algorithms for ranking web pages was proposed in [16]. PageRank is one of the most important algorithms developed for search engines used by Google, with the aim of scoring web-pages based on their relevance. Each webpage that has hyperlinks to other webpages is considered a voter, while the voter's preferences for candidates is expressed via the hyperlinks. When a hyperlink to a webpage is not present, it is assumed that the voter does not support the given candidate webpage. Although the exact implementation details of PageRank are not known, it is widely assumed that the graph of webpages is endowed with near-uniform transition probabilities. The ranking of the webpages is obtained by computing the stationary probabilities of the chain, and ordering the pages according to the values of the stationary probabilities.

This idea can be easily adapted to the rank aggregation problem. In such an adaptation, the states of a Markov chain correspond to the candidates and the transition probabilities are functions of the votes. Dwork et al. [4], [16] proposed four different ways for computing the transition probabilities from the votes. Below, we describe the method that is most suitable for our problem and provide a generalization of the algorithm for aggregation with the weighted Kendall distance.

Consider a Markov chain with states indexed by the candidates. Let P denote the transition probability matrix of the Markov chain, with P_{ij} denoting the probability of going from state (candidate) i to state j. In [16], the transition probabilities are evaluated as

$$P_{ij} = \frac{1}{m} \sum_{\sigma \in \Sigma} P_{ij}(\sigma),$$

where

$$P_{ij}(\sigma) = \begin{cases} \frac{1}{n}, & \text{if } \sigma^{-1}(j) < \sigma^{-1}(i), \\ 1 - \frac{\sigma^{-1}(i)-1}{n}, & \text{if } i = j, \\ 0, & \text{if } \sigma^{-1}(j) > \sigma^{-1}(i). \end{cases}$$

Our Markov chain model for weighted Kendall distance is similar, with a modification that includes incorporating transposition weights into the transition probabilities. To accomplish this task, we proceed as follows.

Let $w_k = \varphi_{(k\,k+1)}$, and let $i_\sigma = \sigma^{-1}(i)$ for candidate $i \in [n]$. We set

$$\beta_{ij}(\sigma) = \max_{l: j_\sigma \le l < i_\sigma} \frac{\sum_{h=l}^{i_\sigma - 1} w_h}{i_\sigma - l} \tag{24}$$

if $j_\sigma < i_\sigma$, $\beta_{ij}(\sigma) = 0$ if $j_\sigma > i_\sigma$, and

$$\beta_{ii}(\sigma) = \sum_{k: k_\sigma > i_\sigma} \beta_{ki}(\sigma).$$

The transition probabilities equal

$$P_{ij} = \frac{1}{m} \sum_{k=1}^{m} P_{ij}(\sigma_k),$$

with

$$P_{ij}(\sigma) = \frac{\beta_{ij}(\sigma)}{\sum_k \beta_{ik}(\sigma)}.$$

Intuitively, the transition probabilities described above may be interpreted as follows. The transition probabilities are obtained by averaging the transition probabilities corresponding to individual votes $\sigma \in \Sigma$. For each vote σ, consider candidates j and k with $j_\sigma = i_\sigma - 1$ and $k_\sigma = i_\sigma - 2$. The probability of going from candidate i to candidate j is proportional to $w_{j_\sigma} = \varphi_{(j_\sigma\, i_\sigma)}$. This implies that if $w_{j_\sigma} > 0$, one moves from candidate i to candidate j with positive probability. Furthermore, larger values for w_{j_σ} result in higher probabilities for moving from i to j.

In the case of candidate k, it seems reasonable to let the probability of transitioning from candidate i to candidate k be proportional to $\frac{w_{j_\sigma} + w_{k_\sigma}}{2}$. However, since k is ranked before j by vote σ, it is natural to require that the probability of moving to candidate k from candidate i be at least as high as the probability of moving to candidate j from candidate i. This reasoning leads to $\beta_{ik}(\sigma) = \max\{w_{j_\sigma}, \frac{w_{j_\sigma} + w_{k_\sigma}}{2}\}$ and motivates using the maximum in (24). Finally, the probability of staying with candidate i is proportional to the sum of the β's from candidates placed below candidate i.

Example 9: Let the votes in Σ consist of $\sigma_1 = (a, b, c)$, $\sigma_2 = (a, b, c)$, and $\sigma_3 = (b, c, a)$, and let $w = (w_1, w_2) = (2, 1)$

Consider the vote $\sigma_1 = (a, b, c)$. We have $\beta_{ba}(\sigma_1) = \frac{w_1}{1} = 2$. Note that if w_1 is large, then β_{ba} is large as well.

In addition, $\beta_{cb}(\sigma_1) = \frac{w_2}{1} = 1$ and

$$\beta_{ca}(\sigma_1) = \max\left\{\frac{w_1 + w_2}{2}, \beta_{cb}(\sigma_1)\right\} = \frac{3}{2}.$$

The purpose of the max function is to ensure that $\beta_{ca}(\sigma_1) \ge \beta_{cb}(\sigma_1)$, which is a natural requirement given that a is ranked before b according to σ_1.

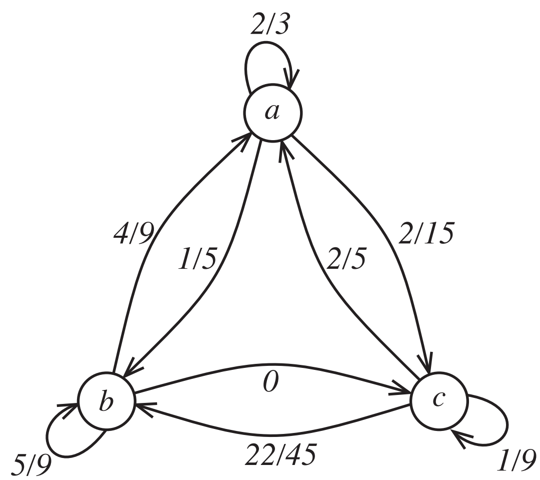


Fig. 8. The Markov chain for Example 9.

Finally, $\beta_{aa}(\sigma_1) = \beta_{ca}(\sigma_1) + \beta_{ba}(\sigma_1) = 2 + \frac{3}{2} = \frac{7}{2}$ and $\beta_{bb}(\sigma_1) = \beta_{cb}(\sigma_1) = 1$. Note that according to the transition probability model, one also has $\beta_{aa}(\sigma_1) \geq \beta_{bb}(\sigma_1)$. This may again be justified by the fact that σ_1 places a higher than b.

Since $\sigma_1 = \sigma_2$, we have

$$P(\sigma_1) = P(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1/3 & 0 \\ 3/5 & 2/5 & 0 \end{pmatrix}.$$

Similar computations yield

$$\beta_{cb}(\sigma_3) = 2, \quad \beta_{ac}(\sigma_3) = 1, \quad \beta_{ab}(\sigma_3) = \frac{3}{2}$$

$$\beta_{aa}(\sigma_3) = 0, \quad \beta_{bb}(\sigma_3) = 2 + \frac{3}{2} = \frac{7}{2}, \quad \beta_{cc}(\sigma_3) = 1,$$

and thus

$$P(\sigma_3) = \begin{pmatrix} 0 & 3/5 & 2/5 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \end{pmatrix}.$$

From the $P(\sigma_1)$, $P(\sigma_2)$, and $P(\sigma_3)$, we obtain

$$P = \frac{P(\sigma_1) + P(\sigma_2) + P(\sigma_3)}{3} = \begin{pmatrix} 2/3 & 1/5 & 2/15 \\ 4/9 & 5/9 & 0 \\ 2/5 & 22/45 & 1/9 \end{pmatrix}.$$

The Markov chain corresponding to P is given in Figure 8. The stationary distribution of this Markov chain is (0.56657, 0.34844, 0.084986) which corresponds to the ranking (a, b, c).

Example 10: The performance of the Markov chain approach described above cannot be easily evaluated analytically, as is the case with any related aggregation algorithm proposed so far.

We hence test the performance of the scheme on examples for which the optimal solutions are easy to evaluate numerically. For this purpose, in what follows, we consider a simple test example, with $m = 11$. The set of votes (rankings) is

$$\Sigma^T = \left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 \\ 2 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 4 & 4 & 4 & 4 & 5 & 5 & 3 & 3 \\ 4 & 4 & 4 & 5 & 5 & 5 & 5 & 3 & 3 & 4 & 4 \\ 5 & 5 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

Note that due to the transpose operator, each column corresponds to a vote, e.g., $\sigma_1 = (1, 2, 3, 4, 5)$.

Let us consider candidates 1 and 2. Using the plurality rule, one would arrive at the conclusion that candidate 1 should be the winner, given that 1 appears most often at the top of the list. Under a number of other aggregation rules, including Kemeny's rule and Borda's method, candidate 2 would be the winner.

Our immediate goal is to see how different weighted distance based rank aggregation algorithms would position candidates 1 and 2. The numerical results regarding this example are presented in Table I. In the table, OPT refers to an optimal solution which was found by exhaustive search, and MC refers to the Markov chain method.

If the weight function is $w = (w_1, \ldots, w_4) = (1, 0, 0, 0)$, where $w_i = \varphi_{(i\, i+1)}$, the optimal aggregate vote clearly corresponds to the plurality winner. That is, the winner is the candidate with most voters ranking him/her as the top candidate. A quick check of Table I reveals that all three methods identify the winner correctly. Note that the ranks of candidates other than candidate 1 obtained by the different methods are different. However this does not affect the distance between the aggregate ranking and the votes.

The next weight function that we consider is the uniform weight function, $w = (1, 1, 1, 1)$. This weight function corresponds to the conventional Kendall τ distance. As shown in Table I, all three methods produce $(2, 3, 4, 5, 1)$, and the aggregates returned by BMLS and MC are optimum.

The weight function $w = (1, 1, 0, 0)$ corresponds to *ranking of the top 2* candidates. OPT and BMLS return 2, 3 as the top two candidates, both preferring 2 to 3. The MC method, however, returns 2, 1 as the top two candidates, with a preference for 2 over 1, and a suboptimal cumulative distance. It should be noted that this may be attributed to the fact the the MC method is not designed to only minimize the average distance: another important factor in determining the winners via the MC method is that winning against strong candidates "makes one strong". In this example, candidate 1 beats the strongest candidate, candidate 2, three times, while candidate 3 beats candidate 2 only twice and this fact seems to be the reason for the MC algorithm to prefer candidate 1 to candidate 3. Nevertheless, the stationary probabilities of candidates 1 and 3 obtained by the MC method are very close to each other, as the vector of probabilities is $(\underline{0.137}, 0.555, \underline{0.132}, 0.0883, 0.0877)$.

The weight function $w = (0, 1, 0, 0)$ corresponds to *identifying the top 2* candidates – i.e., it is not important which candidate is the first and which is the second. The OPT and BMLS identify $\{2, 3\}$ as the top two candidates.

The MC method returns the stationary probabilities $(0, 1, 0, 0, 0)$ which means that candidate 2 is an absorbing state in the Markov chain. This occurs because candidate 2 is ranked first or second by all voters. The existence of absorbing states is a drawback of the Markov chain methods. One solution is to remove 2 from the votes and re-apply MC. The MC method in this case results in the stationary distribution $(p(1), p(3), p(4), p(5)) = (0.273, 0.364, 0.182, 0.182)$,

TABLE I

THE AGGREGATE RANKINGS AND THE AVERAGE DISTANCE OF THE AGGREGATE RANKING FROM THE VOTES FOR DIFFERENT WEIGHT FUNCTIONS w. FOR THE RANKING MARKED BY A *, SEE THE COMMENT IN THE TEXT

Method	Aggregate ranking and average distance			
	$w=(1,0,0,0)$	$w=(1,1,1,1)$	$w=(1,1,0,0)$	$w=(0,1,0,0)$
OPT	$(\underline{1},4,3,2,5)$, 0.7273	$(2,3,4,5,1)$, 2.3636	$\left(\underline{2,3},4,5,1\right)$, 1.455	$\left(\underline{3,2},5,4,1\right)$, 0.636
BMLS	$(\underline{1},2,3,4,5)$, 0.7273	$(2,3,4,5,1)$, 2.3636	$\left(\underline{2,3},1,5,4\right)$, 1.455	$\left(\underline{2,3},1,5,4\right)$, 0.636
MC	$(\underline{1},2,5,4,3)$, 0.7273	$(2,3,4,5,1)$, 2.3636	$\left(\underline{2,1},3,4,5\right)$, 1.546	$\left(\underline{2,3},1,4,5\right)^*$, 0.636

which gives us the ranking $(3,1,4,5)$. Together with the fact that candidate 2 is the strongest candidate, we obtain the ranking $(2,3,1,4,5)$.

APPENDIX

A. Proof of Lemma 6

To prove Lemma 6, we first prove its special case in the following lemma.

Lemma 5: For a non-negative weight function φ and a transposition $(a\,b) \in \mathbb{S}_n$, we have

$$\mathsf{d}_\varphi((a\,b),e) \le 2\mathrm{wt}(p^*_\varphi(a,b)).$$

Furthermore, if φ is a metric weight function, then $\mathsf{d}_\varphi((a\,b),e) \le \mathrm{wt}(p^*_\varphi(a,b))$.

Proof: Consider a path $p=(v_0=a,v_1,\dots,v_{|p|}=b)$ from a to b in $\mathcal{K}_\varphi$. We have

$$\begin{aligned}(a\,b) = {}&(v_0\,v_1)\,(v_1\,v_2)\cdots\left(v_{|p|-2}\,v_{|p|-1}\right)\\ &\left(v_{|p|-1}\,v_{|p|}\right)\left(v_{|p|-2}\,v_{|p|-1}\right)\cdots(v_1\,v_2)\,(v_0\,v_1)\,.\end{aligned}$$

From the triangle inequality and the left-invariance of d_φ,

$$\begin{aligned}\mathsf{d}_\varphi((a\,b),e) &\le 2\sum_{i=1}^{|p|}\varphi_{(v_{i-1}\,v_i)} - \varphi_{(v_{|p|-1}\,v_{|p|})}\\ &= 2\mathrm{wt}(p) - \varphi_{(v_{|p|-1}\,v_{|p|})}\\ &\le 2\mathrm{wt}(p).\end{aligned}$$

Since p is an arbitrary path from a to b in $\mathcal{K}_\varphi$, we have

$$\mathsf{d}_\varphi\,((a\,b),e) \le 2\mathrm{wt}(p^*_\varphi(a,b)),$$

and this proves the first claim.

Now, assume that φ is a metric weight function and consider the path $p=(v_0,v_1,\dots,v_{|p|})$ from $v_0=a$ to $v_{|p|}=b$. From (19),

$$\begin{aligned}\varphi_{(a\,b)} &= \varphi_{(v_0\,v_{|p|})} \le \varphi_{(v_0\,v_1)} + \varphi_{(v_1\,v_{|p|})}\\ &\le \varphi_{(v_0\,v_1)} + \varphi_{(v_1\,v_2)} + \varphi_{(v_2\,v_{|p|})}\\ &\le \cdots\\ &\le \sum_{i=1}^{|p|}\varphi_{(v_{i-1}\,v_i)}\\ &= \mathrm{wt}(p).\end{aligned}$$

Since p is arbitrary, we have

$$\mathsf{d}_\varphi((a\,b),e) \le \varphi_{(a\,b)} \le \mathrm{wt}(p^*_\varphi(a,b)).$$

This completes the proof of the lemma. □

While Lemma 5 suffices to prove Lemma 6, we remark that one may prove a slightly stronger result, presented in our companion paper [33],

$$\mathsf{d}_\varphi((a\ b),e) = \min_{p=(v_0,\dots,v_{|p|})}\left(2\mathrm{wt}(p) - \max_{0\le i<|p|}\varphi_{(v_i\,v_{i+1})}\right),$$

where $v_0=a$ and $v_{|p|}=b$. The proof is based on significantly more involved techniques that are beyond the scope of this paper.

Lemma 6: For a weight function φ and for $\pi,\sigma \in \mathbb{S}_n$, we have $\mathsf{d}_\varphi(\pi,\sigma) \le 2D_\varphi(\pi,\sigma)$. If φ is a metric weight function, the bound may be improved to $\mathsf{d}_\varphi(\pi,\sigma) \le D_\varphi(\pi,\sigma)$.

Proof: To prove the first claim, it suffices to show that $\mathsf{d}_\varphi(\pi,e) \le 2D_\varphi(\pi,e)$ since both d_φ and D_φ are left-invariant.

Let $\{c_1,c_2,\dots,c_k\}$ be the cycle decomposition of π. We have, from the triangle inequality and the left-invariance property of d_φ, that

$$\mathsf{d}_\varphi(\pi,e) \le \sum_{i=1}^{k}\mathsf{d}_\varphi(c_i,e),$$

and, from the definition of D_φ, that

$$D_\varphi(\pi,e) = \sum_{i=1}^{k}D_\varphi(c_i,e).$$

Hence, we only need to prove that

$$\mathsf{d}_\varphi(c,e) \le 2D_\varphi(c,e) \tag{25}$$

for a single cycle $c=(a_1\,a_2\,\cdots\,a_{|c|})$, where $|c|$ is the length of c.

Since c may be written as

$$c = (a_1\,a_2)(a_2\,a_3)\cdots(a_{|c|-1}\,a_{|c|}),$$

we have

$$\begin{aligned}\mathsf{d}_\varphi(c,e) &\le \sum_{i=1}^{|c|-1}\mathsf{d}_\varphi((a_i\,a_{i+1}),e)\\ &\overset{(a)}{\le} \sum_{i=1}^{|c|-1}2\mathrm{wt}(p^*_\varphi(a_i,a_{i+1}))\\ &\le \sum_{i=1}^{|c|}2\mathrm{wt}(p^*_\varphi(a_i,c(a_i)))\\ &\le 2D_\varphi(c,e)\end{aligned}$$

where (a) follows from Lemma 5. The proof of the second claim is similar except that the factor 2 is not needed in inequality (a) and the following inequalities. □

B. Computing the Weight Functions With Two Identical Non-Zero Weights

The goal is to find the weighted Kendall distance $\mathsf{d}_\varphi(\pi, e)$ with the weight function of (14), for an arbitrary $\pi \in \mathbb{S}_n$. For this purpose, let $R_1 = \{1, \dots, a\}$, $R_2 = \{a+1, \dots, b\}$, and $R_3 = \{b+1, \dots, n\}$, and define

$$N_{ij}^{\pi} = |\{k \in R_j : \pi^{-1}(k) \in R_i\}|, \quad i, j \in \{1, 2, 3\}.$$

That is, N_{ij}^{π} is the number of elements whose ranks in π belong to the set R_i and whose ranks in e belong to the set R_j. A sequence of transpositions that transforms π into e moves the N_{ij}^{π} elements of $\{k \in R_j : \pi^{-1}(k) \in R_i\}$ from R_i to R_j. Furthermore, note that any transposition that swaps two elements with ranks in the same region $R_i, i \in [3]$, has weight zero, while for any transposition τ_l that swaps an element ranked in R_1 with an element ranked in R_2 or swaps an element ranked in R_2 with an element ranked in R_3, we have $\mathsf{d}_\varphi(\tau_l, e) = 1$.

It is straightforward to see that $\sum_j N_{ij}^{\pi} = \sum_j N_{ji}^{\pi}$. In particular, $N_{12}^{\pi} + N_{13}^{\pi} = N_{21}^{\pi} + N_{31}^{\pi}$ and $N_{31}^{\pi} + N_{32}^{\pi} = N_{13}^{\pi} + N_{23}^{\pi}$.

We show next that

$$\mathsf{d}_\varphi(\pi, e) = \begin{cases} 2N_{13}^{\pi} + N_{12}^{\pi} + N_{23}^{\pi}, & \text{if } N_{21}^{\pi} \geq 1 \text{ or } N_{23}^{\pi} \geq 1, \\ 2N_{13}^{\pi} + 1, & \text{if } N_{21}^{\pi} = N_{23}^{\pi} = 0. \end{cases}$$

Note that, from Proposition 2, we have

$$\mathsf{d}_\varphi(\pi, e) \geq \frac{1}{2} D_\varphi(\pi, e) = 2N_{13}^{\pi} + N_{12}^{\pi} + N_{23}^{\pi}. \qquad (26)$$

Suppose that $N_{21}^{\pi} \geq 1$ or $N_{23}^{\pi} \geq 1$. We find a transposition τ_l, with $\mathsf{d}_\varphi(\tau_l, e) = 1$, such that $\pi' = \pi\tau_l$ satisfies $D_\varphi(\pi', e) = D_\varphi(\pi, e) - 2$, and at least one of the following conditions:

$$\begin{cases} N_{21}^{\pi'} \geq 1, \\ \text{or} \\ N_{23}^{\pi'} \geq 1, \\ \text{or} \\ \pi' = e. \end{cases} \qquad (27)$$

Applying the same argument repeatedly, and using the triangle inequality proves that $\mathsf{d}_\varphi(\pi, e) \leq \frac{1}{2} D_\varphi(\pi, e)$ if $N_{21}^{\pi} \geq 1$ or $N_{23}^{\pi} \geq 1$. This, along with (26), shows that $\mathsf{d}_\varphi(\pi, e) = \frac{1}{2} D_\varphi(\pi, e)$ if $N_{21}^{\pi} \geq 1$ or $N_{23}^{\pi} \geq 1$.

First, suppose that $N_{21}^{\pi} \geq 1$ and $N_{23}^{\pi} \geq 1$. It then follows that $N_{12}^{\pi} \geq 1$ or $N_{32}^{\pi} \geq 1$. Without loss of generality, assume that $N_{12}^{\pi} \geq 1$. Then τ_l can be chosen such that $N_{12}^{\pi'} = N_{12}^{\pi} - 1$ and $N_{21}^{\pi'} = N_{21}^{\pi} - 1$. We have $D_\varphi(\pi', e) = D_\varphi(\pi, e) - 2$, and since $N_{23}^{\pi} \geq 1$, condition (27) holds.

Next, suppose $N_{21}^{\pi} \geq 1$ and $N_{23}^{\pi} = 0$. If $N_{13}^{\pi} \geq 1$, choose τ_l such that

$$N_{21}^{\pi'} = N_{21}^{\pi} - 1, \quad N_{23}^{\pi'} = 1, \quad N_{13}^{\pi'} = N_{13}^{\pi} - 1,$$

where $\pi' = \pi\tau_l$. Since $N_{23}^{\pi'} = 1$, condition (27) is satisfied. If $N_{13}^{\pi} = 0$, then $N_{31}^{\pi} = N_{32}^{\pi} = 0$, and thus $N_{12}^{\pi} = N_{21}^{\pi} \geq 1$. In this case, we choose τ_l such that $N_{21}^{\pi'} = N_{12}^{\pi'} = N_{12}^{\pi} - 1$. As a result, we have either $N_{21}^{\pi'} \geq 1$ or $\pi' = e$. Hence, condition (27) is satisfied once again. Note that in both cases, for $N_{13}^{\pi} = 0$ as well as for $N_{13}^{\pi} \geq 1$, we have $D_\varphi(\pi', e) = D_\varphi(\pi, e) - 2$.

The proof for the case $N_{23}^{\pi} \geq 1$ and $N_{21}^{\pi} = 0$ follows along similar lines.

If $N_{21}^{\pi} = N_{23}^{\pi} = 0$, it can be verified by inspection that for every transposition τ_l with $\mathsf{d}_\varphi(\tau_l, e) = 1$, we have $D_\varphi(\pi\tau_l, e) \geq D_\varphi(\pi, e)$. Hence, the inequality in (26) cannot be satisfied with equality, which implies that $\mathsf{d}_\varphi(\pi, e) \geq 2N_{13}^{\pi} + 1$. Choose a transposition τ_l with $\mathsf{d}_\varphi(\tau_l, e) = 1$ such that

$$N_{13}^{\pi'} = N_{13}^{\pi} - 1, \quad N_{12}^{\pi'} = 1, \quad N_{23}^{\pi'} = 1.$$

where $\pi' = \pi\tau_l$. We have

$$\mathsf{d}_\varphi(\pi, e) \leq \mathsf{d}_\varphi(\tau_l, e) + \mathsf{d}_\varphi(\pi', e) = 1 + 2N_{13}^{\pi}.$$

This, along with $\mathsf{d}_\varphi(\pi, e) \geq 2N_{13}^{\pi} + 1$, completes the proof.

C. The Average Kendall and Weighted Kendall Distances

The Kendall τ distance between two rankings may be viewed in the following way: each pair of candidates on which the two rankings disagree contribute one unit to the distance between the rankings. Owing to Algorithm 1, the weighted Kendall distance with a decreasing weight function can be regarded in a similar manner: each pair of candidates on which the two rankings disagree contributes $\varphi_{(s\ s+1)}$, for some s, to the distance between the rankings.

Consider a pair a and b such that $\pi^{-1}(b) < \pi^{-1}(a)$ and $\sigma^{-1}(a) < \sigma^{-1}(b)$. In Algorithm 1, there exists a transposition $\tau_t^\star = (s\ s+1)$ that swaps a and b where

$$s = \pi^{-1}(b) + \left|\left\{k : \sigma^{-1}(k) < \sigma^{-1}(a), \pi^{-1}(k) > \pi^{-1}(b)\right\}\right|,$$

that is, s equals $\pi^{-1}(b)$ plus the number of elements that appear before a in σ and after b in π. It is not hard to see that s can also be written in a way that is symmetric with respect to π and σ, as

$$\begin{aligned} s &= \pi^{-1}(b) + \sigma^{-1}(a) \\ &\quad - \left|\left\{k : \pi^{-1}(k) < \pi^{-1}(b), \sigma^{-1}(k) < \sigma^{-1}(a)\right\}\right| - 1 \\ &= n - 1 - \left|\left\{k : \pi^{-1}(k) > \pi^{-1}(b), \sigma^{-1}(k) > \sigma^{-1}(a)\right\}\right|. \end{aligned}$$

As an example, consider $\varphi_{(i\ i+1)} = n - i$. Then, $\mathsf{d}_\varphi(\pi, \sigma)$ equals

$$\begin{aligned} &\sum_{(b,a)\in\mathscr{I}(\pi,\sigma)} \left(1 + \left|\left\{k : \pi^{-1}(k) > \pi^{-1}(b), \sigma^{-1}(k) > \sigma^{-1}(a)\right\}\right|\right) \\ &= K(\pi, \sigma) + \\ &\sum_{(b,a)\in\mathscr{I}(\pi,\sigma)} \left|\left\{k : \pi^{-1}(k) > \pi^{-1}(b), \sigma^{-1}(k) > \sigma^{-1}(a)\right\}\right|, \end{aligned}$$

where $\mathscr{I}(\pi, \sigma)$ is the set of ordered pairs (b, a) such that $\pi^{-1}(b) < \pi^{-1}(a)$ and $\sigma^{-1}(a) < \sigma^{-1}(b)$. Note that the weighted Kendall distance d_φ equals the Kendall τ distance plus a sum that captures the influence of assigning higher importance to the top positions of the rankings.

We now compute the expected value of the weighted Kendall distance with weight function φ between the identity permutation and a randomly and uniformly chosen permutation $\pi \in \mathbb{S}_n$. For $1 \le a < b \le n$ and $s \in [n-1]$, let X_{ab}^s be an indicator variable that equals one if and only if $\pi^{-1}(a) > \pi^{-1}(b)$ and

$$\left|\left\{k > a : \pi^{-1}(k) > \pi^{-1}(b)\right\}\right| = n - 1 - s.$$

The expected distance between the two permutations equals

$$E[\mathsf{d}_\varphi(\pi, e)] = \sum_{s=1}^{n-1} \varphi_{(s\ s+1)} \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} E\left[X_{ab}^s\right]. \tag{28}$$

By the definition of X_{ab}^s, $E\left[X_{ab}^s\right]$ equals the probability of the event that $n-1-s$ elements of $\{a+1, \ldots, n\}\backslash\{b\}$ and a appear after b in π. There are $\binom{n-a-1}{n-s-1}$ ways to choose $n-s-1$ elements from $\{a+1, \ldots, n\}\backslash\{b\}$, $\binom{n}{a-1}(a-1)!$ ways to assign positions to the elements of $\{1, 2, \ldots, a-1\}$, $(s-a)!$ ways to arrange the $s-a$ elements of $\{a+1, \ldots, n\}\backslash\{b\}$ that appear before b, and $(n-s)!$ ways to arrange a and the $n-1-s$ elements of $\{a+1, \ldots, n\}\backslash\{b\}$ that appear after b. Hence,

$$\begin{aligned} E\left[X_{ab}^s\right] &= \frac{1}{n!}\binom{n-a-1}{n-s-1}\binom{n}{a-1}(a-1)!(s-a)!(n-s)! \\ &= \frac{n-s}{(n-a+1)(n-a)}, \end{aligned}$$

for $1 \le a \le s$, and $E\left[X_{ab}^s\right] = 0$ for $a > s$. By using these expressions and (28), we obtain

$$\begin{aligned} E[\mathsf{d}_\varphi(e, \pi)] &= \sum_{s=1}^{n-1} \varphi_{(s\ s+1)} \sum_{a=1}^{s} \frac{n-s}{n-a+1} \\ &= \sum_{s=1}^{n-1} \varphi_{(s\ s+1)} (n-s)(H_n - H_{n-s}), \end{aligned}$$

where $H_i = \sum_{l=1}^{i} \frac{1}{l}$. Indeed, for $\varphi_{(s\,s+1)} = 1, s \in [n-1]$, we recover the well known result that

$$\begin{aligned} E[\mathsf{d}_\varphi(e, \pi)] &= \sum_{s=1}^{n-1} (n-s)(H_n - H_{n-s}) \\ &= \sum_{k=1}^{n-1} k(H_n - H_k) \\ &= \frac{1}{2}\binom{n}{2}. \end{aligned}$$

For $\varphi_{(s\ s+1)} = n - s$, the average distance equals

$$\begin{aligned} E[\mathsf{d}_\varphi(e, \pi)] &= \sum_{s=1}^{n-1} (n-s)^2(H_n - H_{n-s}) \\ &= \sum_{k=1}^{n-1} k^2(H_n - H_k) \\ &= \frac{1}{2}\binom{n}{2} + \frac{2}{3}\binom{n}{3}. \end{aligned}$$

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References

[1] M. Kendall, *Rank Correlation Methods*. London, U.K.: Griffin, 1948.

[2] J. Marden, *Analyzing and Modeling Rank Data* (Monographs on Statistics and Applied Probability). London, U.K.: Chapman & Hall, 1995. [Online]. Available: http://books.google.com/books?id=Dp24H21QHNoC

[3] W. H. Kruskal and W. A. Wallis, "Use of ranks in one-criterion variance analysis," *J. Amer. Statist. Assoc.*, vol. 47, no. 260, pp. 583–621, Dec. 1952.

[4] C. Dwork, R. Kumar, M. Naor, and D. Sivakumar. (2001). *Rank Aggregation Revisited*. [Online]. Available: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.23.5118

[5] E. Yilmaz, J. A. Aslam, and S. Robertson, "A new rank correlation coefficient for information retrieval," in *Proc. 31st Annu. Int. ACM SIGIR Conf. Res. Develop. Inf. Retr.*, 2008, pp. 587–594.

[6] J. G. Kemeny, "Mathematics without numbers," *Daedalus*, vol. 88, no. 4, pp. 577–591, 1959.

[7] H. P. Young, "Social choice scoring functions," *SIAM J. Appl. Math.*, vol. 28, no. 4, pp. 824–838, 1975.

[8] M. Regenwetter, B. Grofman, A. A. J. Marley, and I. Tsetlin, *Behavioral Social Choice: Probabilistic Models, Statistical Inference, and Applications*. Cambridge, U.K.: Cambridge Univ. Press, 2006.

[9] A. Jiang, M. Schwartz, and J. Bruck, "Error-correcting codes for rank modulation," in *Proc. IEEE Int. Symp. Inf. Theory*, Toronto, ON, Canada, Jul. 2008, pp. 1736–1740.

[10] F. Farnoud, V. Skachek, and O. Milenkovic, "Error-correction in flash memories via codes in the Ulam metric," *IEEE Trans. Inf. Theory*, vol. 59, no. 5, pp. 3003–3020, May 2013.

[11] D. M. Pennock, E. Horvitz, and C. L. Giles, "Social choice theory and recommender systems: Analysis of the axiomatic foundations of collaborative filtering," in *Proc. Nat. Conf. Artif. Intell.*, Austin, TX, USA, 2000, pp. 729–734.

[12] M. Kim, F. Raisali, F. Farnoud, and O. Milenkovic, "Gene prioritization via weighted Kendall rank aggregation," in *Proc. IEEE 5th Int. Workshop Comput. Adv. Multi-Sensor Adapt. Process. (CAMSAP)*, Dec. 2013, pp. 184–187.

[13] J. L. Herlocker, J. A. Konstan, L. G. Terveen, and J. T. Riedl, "Evaluating collaborative filtering recommender systems," *ACM Trans. Inf. Syst.*, vol. 22, no. 1, pp. 5–53, Jan. 2004. [Online]. Available: http://doi.acm.org/10.1145/963770.963772

[14] A. Mazumdar, A. Barg, and G. Zemor, "Constructions of rank modulation codes," *IEEE Trans. Inf. Theory*, vol. 59, no. 2, pp. 1018–1029, Feb. 2013.

[15] H. Young and A. Levenglick, "A consistent extension of Condorcet's election principle," *SIAM J. Appl. Math.*, vol. 35, no. 2, pp. 285–300, Sep. 1978. [Online]. Available: http://epubs.siam.org/doi/abs/10.1137/0135023

[16] C. Dwork, R. Kumar, M. Naor, and D. Sivakumar, "Rank aggregation methods for the web," in *Proc. 10th Int. Conf. World Wide Web*, 2001, pp. 613–622.

[17] I. Salama and D. Quade, "A nonparametric comparison of two multiple regressions by means of a weighted measure of correlation," *Commun. Statist.-Theory Methods*, vol. 11, no. 11, pp. 1185–1195, 1982.

[18] G. S. Shieh, "A weighted Kendall's tau statistic," *Statist. Probab. Lett.*, vol. 39, no. 1, pp. 17–24, Jul. 1998.

[19] R. Kumar and S. Vassilvitskii, "Generalized distances between rankings," in *Proc. 19th Int. Conf. World Wide Web*, Raleigh, NC, USA, 2010, pp. 571–580.

[20] W. D. Cook and M. Kress, "Ordinal ranking with intensity of preference," *Manage. Sci.*, vol. 31, no. 1, pp. 26–32, Jan. 1985.

[21] D. Sculley, "Rank aggregation for similar items," in *Proc. 7th SIAM Int. Conf. Data Mining*, Minneapolis, MN, USA, 2007.

[22] F. Schalekamp and A. van Zuylen, "Rank aggregation: Together we're strong," in *Proc. 11th Algorithm Eng. Experim. (ALENEX)*, 2009, pp. 38–51.

[23] J.-C. de Borda, "Mémoire sur les élections au scrutin," *Histoire de l'Acad. Roy. des Sci.*, pp. 657–665, 1781.
[24] P. Diaconis, *Group Representations in Probability and Statistics*, vol. 11. Hayward, CA, USA: Institute of Mathematical Statistics, 1988.
[25] J. G. Kemeny and J. L. Snell, *Mathematical Models in the Social Sciences*. Boston, MA, USA: Ginn, 1962.
[26] B. Carterette, "On rank correlation and the distance between rankings," in *Proc. 32nd Int. AVM SIGIR Conf. Res. Develop. Inf. Retr.*, 2009, pp. 436–443.
[27] P. H. Lee and P. L. H. Yu, "Distance-based tree models for ranking data," *Comput. Statist. Data Anal.*, vol. 54, no. 6, pp. 1672–1682, Jun. 2010.
[28] F. Hassanzadeh, "Distances on rankings: From social choice to flash memories," Ph.D. dissertation, Dept. Elect. & Comput. Eng., Univ. Illinois Urbana-Champaign, Champaign, IL, USA, May 2013. [Online]. Available: http://hdl.handle.net/2142/44268
[29] M. Sun, G. Lebanon, and K. Collins-Thompson, "Visualizing differences in web search algorithms using the expected weighted Hoeffding distance," in *Proc. 19th Int. Conf. World Wide Web*, Raleigh, NC, USA, 2010, pp. 931–940. [Online]. Available: http://doi.acm.org/10.1145/1772690.1772785
[30] R. Fagin, R. Kumar, and D. Sivakumar, "Comparing top *k* lists," *SIAM J. Discrete Math.*, vol. 17, no. 1, pp. 134–160, 2003.
[31] K. Järvelin and J. Kekäläinen, "Cumulated gain-based evaluation of IR techniques," *ACM Trans. Inf. Syst.*, vol. 20, no. 4, pp. 422–446, Oct. 2002.
[32] S. Vigna. (Apr. 2014). "A weighted correlation index for rankings with ties." [Online]. Available: http://arxiv.org/abs/14o4.3325v2
[33] F. Farnoud and O. Milenkovic, "Sorting of permutations by cost-constrained transpositions," *IEEE Trans. Inf. Theory*, vol. 58, no. 1, pp. 3–23, Jan. 2012.
[34] S. Angelov, K. Kunal, and A. McGregor, "Sorting and selection with random costs," in *LATIN: Theoretical Informatics*. Berlin, Germany: Springer-Verlag, 2008, pp. 48–59.
[35] Economist Intelligence Unit, (Aug. 2012) *A Summary of the Liveability Ranking and Overview [White Paper]*. http://www.eiu.com/site_info.asp?info_name=The_Global_Liveability_Report
[36] *The Changing Face of SERPs: Organic Click Through Rate [White Paper]*, Optify Inc, Seattle, WA, USA, 2011.
[37] M. Deza and T. Huang, "Metrics on permutations, a survey," *J. Combinat., Inf., Syst. Sci.*, vol. 23, no. 1–4, pp. 173–185, 1998.
[38] W. R. Knight, "A computer method for calculating Kendall's tau with ungrouped data," *J. Amer. Statist. Assoc.*, vol. 61, no. 314, pp. 436–439, 1966. [Online]. Available: http://www.jstor.org/stable/2282833
[39] T. M. Chan and M. Pătraşcu, "Counting inversions, offline orthogonal range counting, and related problems," in *Proc. 21st Annu. ACM-SIAM Symp. Discrete Algorithms*, Philadelphia, PA, USA, 2010, pp. 161–173. [Online]. Available: http://dl.acm.org/citation.cfm?id=1873601.1873616
[40] M. L. Fredman and R. E. Tarjan, "Fibonacci heaps and their uses in improved network optimization algorithms," *J. ACM*, vol. 34, no. 3, pp. 596–615, Jul. 1987.
[41] K. J. Arrow, *Social Choice and Individual Values*. New Haven, CT, USA: Yale Univ. Press, 1963.
[42] J. Hodge and R. Klima, *The Mathematics of Voting and Elections: A Hands-on Approach* (Mathematical World), vol. 22. Providence, RI, USA: AMS, 2005.
[43] F. Farnoud, B. Touri, and O. Milenkovic, "Nonuniform vote aggregation algorithms," in *Proc. Int. Conf. Signal Process. Commun. (SPCOM)*, Bangalore, India, Jul. 2012, pp. 1–5.

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