On the Multimessage Capacity Region for Undirected Ring Networks

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***Abstract*—The "Japanese" theorem is extended to multiple multicast sessions in an arbitrary network to characterize the routing capacity region by the intersection of an infinite collection of half-spaces. An elimination technique is developed to simplify this infinite description into a finite one based upon the shortest routing paths and trees in the network graph. This result is used as a step in providing the capacity regions for two multimessage multicast problems on undirected ring networks; in the first case only unicast and broadcast sessions are considered, and in the second case multicast sessions where the source and destination vertices form lines of adjacent vertices are studied. Network coding is generally necessary to achieve network capacity, but for our multimessage multicast problems, new arguments are used to demonstrate that routing can achieve network coding bounds.**

***Index Terms*—Japanese theorem, multicast sessions, network capacity, ring networks, routing.**

I. INTRODUCTION

A fundamental problem of network information theory is to compute the capacity region of a network with multiple simultaneous communication sessions. A *session* refers to the communication from a source vertex to a set of destination vertices. A *unicast* session has a single destination, a *multicast* session has at least two destinations, and a *broadcast* session is the special case of a multicast session where all of the vertices in the network except the source are destinations. Before the seminal work by Ahlswede *et al.* [1] the throughput of a wireline network was generally studied in terms of *routing* protocols. Routing allows vertices to only receive and forward the messages of different sessions; it does not permit more complex operations on messages. Reference [1] showed that the maximum throughput of a single multicast sesson is the capacity of the minimum *cut* from the source to all destinations; a cut is a set of edges which, when removed from the network, leave at least one destination in a different component of the resulting network from the source. This capacity is not generally achievable by routing. Sometimes *network coding* is necessary; i.e., vertices are allowed to send some function of the data that they receive from other vertices in the network and their own messages along their outgoing links. We will, respectively, refer to the *network coding capacity region* or *routing capacity region* of a network as the set of achievable rates among the concurrent sessions using network coding or routing protocols.

Finding the routing capacity region of a network is equivalent to solving the problem of fractionally packing *Steiner* trees in the network graph; a Steiner tree is a tree which connects the source of a session to all destinations of that session. The routing capacity region is an inner bound to the network coding capacity region for the same communication problem. Li *et al.* [18] considered *undirected networks* in which communication links are bidirectional, and the total flow in both directions is limited by the capacity of the link. They showed that for a single multicast session the "Steiner strength" of an undirected network provides an upper bound to the network coding capacity which is at most twice the routing capacity for the same problem. This bound does not extend to graphs with multiple multicast sessions, and there is often a gap between the routing capacity region and the best information theoretic outer bounds on the network coding capacity region such as those offered by bidirected cut set bounds [13] or progressive d-separating edge set (*PdE*) bounds [14], [15].

In this paper, we develop a new technique which leads to the tight characterization of the routing capacity region of an arbitrary network. The routing capacity region of networks with multiple sessions can be formulated as a system of linear inequalities in the (total) rates and the partial rates; each partial rate is the portion of the flow of a session that is routed along a specific Steiner tree. This initial formulation is not the solution to our problem because we do not want the partial rates as part of our description. Fourier–Motzkin elimination [24] is a procedure to project the set of solutions of a general set of linear inequalities to a subset of the variables; this can in principle be applied to the initial formulation of our problem to obtain the routing capacity region, but this approach would be complex. Our strategy is different. The "Japanese" theorem of [9],

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[20] describes the routing capacity region of networks with multiple unicast sessions and no multicast sessions as an infinite set of inequalities. Each inequality corresponds to a different vector of "distances" assigned to each edge in the network. Each edge distance can be chosen as an arbitrary nonnegative integer, and this is why there are initially infinitely many inequalities to consider. We extend the Japanese theorem to networks supporting multiple multicast sessions, and this again results in an infinite description of the capacity region. We next consider the boundary points of the polyhedral solution and develop a novel algorithmic technique to find the finite set of necessary and sufficient inequalities among the infinite set of Japanese theorem inequalities. More specifically, our "inequality elimination" technique checks the redundancy of any inequality in defining the routing capacity region.

A second focus of this paper is the network coding capacity region of undirected ring networks. An undirected ring network is a mathematical model consisting of an undirected graph with the topology of a cycle; the vertices of the graph communicate via edges, and the sum of the flow along the two directions of an edge is bounded by its capacity. In the past two to three decades, the increasing need for high-bandwidth, reliable, and potentially long-distance communication systems caused by various high-demand and real-time applications and services resulted in the extensive deployment of communication networks based on SONET/SDH rings (see, e.g., [2], [6], [25], and [28]). Because of their commercial importance, ring networks have been widely studied. The routing capacity region of multiple unicast sessions in undirected ring networks was first derived by Okamura and Seymour [19] as the special case of a more general result for planar graphs and later by Vachani *et al.* in [28] with a different method. The necessary and sufficient conditions for a collection of multiple unicast sessions to be feasible by routing is for the total rate across every cut in the network to be bounded from above by the capacity of the cuts. For ring networks it is known that the cut set bounds offer a tight characterization in the special case of multiple unicast sessions [8], [13], [22]. These bounds correspond to the set of Japanese theorem inequalities with exactly two nonzero edge distances, both of which are equal to one. We focus on the two special cases where

- the source and destination vertices of each communication session form a string of adjacent vertices, and
- each session is either a broadcast or a unicast session.

In these cases we derive the routing capacity region and use a new argument to show that routing is rate-optimal; i.e., the network coding capacity region is no larger than the routing capacity region. We use our inequality elimination technique to prove that for the two special cases of the multiple multicast problem that we study here, we can restrict our attention to edge distances in the set $\{0, 1\}$. The next step of our analysis is to show that the network coding capacity region of each of these communication problems is identical to its routing capacity region. Our outer bounds on the network coding capacity region are based on a new analysis which extracts common information from edge cuts in order to increase some of the coefficients of the rates that appear in the inequalities.

While our focus is on the derivation of capacity regions, earlier work has considered other aspects of deploying network coding in ring networks. For example, the authors of [7] investigated the benefits of network coding for saving energy in a number of broadcast wireless network topologies including rings. They showed that low complexity network coding schemes double the energy efficiency of ring networks.

A different aspect of ring networks is considered in [23], which studies packet-switched wavelength division multiplexing (WDM) on unidirectional and bidirectional ring networks. In this model the total capacity of the ring is divided into different wavelengths, and each node has access to a specific wavelength for receiving or sending packets. The authors of [23] consider a destination stripping protocol, where packets are removed from the ring by their destinations upon the completion of transmission. The authors investigate various statistics of the routing capacity region for a probabilistic multiple multicast problem in which the probability that a particular session is among the set of sessions to be supported is proportional to its number of destinations.

The remainder of this paper is organized as follows: In Section II we generalize the Japanese theorem to multiple multicast networks. We next develop our elimination technique to reduce the infinite description of the routing capacity region into a finite one. In Section III, we consider two classes of communication problems on undirected ring networks and prove that in these cases we need only consider edge distances in $\{0, 1\}$. In Section IV, we establish that the routing bounds also apply when network coding is permitted and conclude that routing is rate-optimal.

II. Routing Capacity Region in Networks

A. The Japanese Theorem

Consider an undirected or directed network $G(V, E)$ in which the edge set is $E = \{e_1, \ldots, e_{|E|}\}$, the vertex set is $V = \{v_1, \ldots, v_{|V|}\}$ where for every set A, $|A|$ denotes the cardinality of the set. Let S denote the set of multicast sessions. Let C_e represent the capacity of edge $e \in E$, i.e., the maximum flow that can pass through edge e. In this paper, we assume that all capacities are rational. A multicast session $s \in S$ with rate R_s is defined by a source vertex $\nu_s \in V$ and a set of destination vertices $D_s \subset V$ each of which receives the messages in session s. The set of trees that span $\nu_s \cup D_s$ in G is denoted by $\mathcal{T}_s$. A feasible routing solution assigns to each spanning tree $T \in \bigcup_s \mathcal{T}_s$ a partial rate $r_T \geq 0$ that satisfies the following two conditions:

1) $\sum_{T \in \mathcal{T}_s} r_T = R_s$ for every $s \in S$
2) $\sum_{s \in S} \sum_{T \in \mathcal{T}_s : e \in T} r_T \leq C_e$ for every $e \in E$.

We call the rate vector $R = (R_1, \ldots, R_{|S|})$ *routing-feasible* if there is a feasible routing solution for it. The "Japanese" theorem of [9], [20] characterizes the set of all routing-feasible rate vectors for an arbitrary network with multiple unicast sessions with an infinite set of linear constraints. We start by the extending the Japanese theorem to the multiple multicast case. Let $\mathbf{a} = (a_1, \ldots, a_{|E|})$ denote a "distance" vector that assigns edge e a nonnegative integral distance a_e. Then for any path or tree T we define its length $L_{\mathbf{a}}(T)$ to be the sum of the distances of the edges in T. Let $\ell_{\mathbf{a}}(s) = \min_{T \in \mathcal{T}_s} L_{\mathbf{a}}(T)$. The Japanese theorem of [9], [20] is as follows.

Theorem 2.1 (The Japanese Theorem): If S is a set of unicast sessions, the polytope $P \subset \mathbb{R}^{|S|}$ of all routing-feasible rate vectors $R = (R_1, \ldots, R_{|S|})$ is shown in (1) at the bottom of the page.

The following result generalizes Theorem 2.1 to include multisession multicast routing.

Theorem 2.2 (The Extended Japanese Theorem): If S is a set of multicast sessions, the polytope $P \subset \mathbb{R}^{|S|}$ of all routing-feasible rate vectors $R = (R_1, \ldots, R_{|S|})$ is also determined by (1).

Theorems 2.1 and 2.2 are both consequences of Farkas' Lemma (see, e.g., [30, Sec. 4.1]). We prove Theorem 2.2 in Appendix I.

Because Fourier–Motzkin elimination results in a finite description of the routing capacity region, it follows that the infinite set of inequalities in Theorem 2.2 contains infinitely many redundant constraints. We next introduce a method to eliminate the redundant constraints.

B. The Reduced Set of Inequalities

An inequality in (1) is said to be *redundant* if it is implied by other inequalities in (1). A *minimal* set of inequalities that defines P is then a subset of inequalities in (1) with no redundant inequality. For distance vector $\mathbf{a}$, we say that rate vector R is on the hyperplane corresponding to $\mathbf{a}$ if $\sum_{s\in S} \ell_{\mathbf{a}}(s)R_s = \sum_{e\in E} a_e C_e$. We have the following result.

Lemma 2.3: A minimal set of inequalities that defines P is unique up to the multiplication of inequalities by positive scalars. Furthermore, if $\mathbf{a}$ and $\mathbf{b}$ are two distance vectors such that every *routing-feasible* rate vector R on the hyperplane corresponding to $\mathbf{a}$ is also on the hyperplane corresponding to $\mathbf{b}$, then the inequality corresponding to $\mathbf{a}$ in (1) is redundant.

Proof: See Appendix II. ■

Lemma 2.4: The routing-feasible rate vector $R \in P$ is on the hyperplane corresponding to $\mathbf{a}$ if and only if

1) For each session $s \in S$ and every $T \in \mathcal{T}_s$, $r_T = 0$ if $L_{\mathbf{a}}(T) > \ell_{\mathbf{a}}(s)$; i.e., session s is routed only along the shortest paths and trees determined by the distance vector $\mathbf{a} = (a_1, \ldots, a_{|E|})$, and
2) $\sum_{s\in S} \sum_{T\in\mathcal{T}_s : e\in T} r_T = C_e$ for every $e \in E$ with $a_e > 0$; i.e., every edge with a nonzero distance is fully utilized.

Proof: To establish necessity, assume that the rate vector $R = (R_1, \ldots, R_{|S|})$ is routable and is on the hyperplane $\sum_{s\in S} \ell_{\mathbf{a}}(s)R_s = \sum_{e\in E} a_e C_e$. For any edge e in the network, the sum of all flows passing through it is at most C_e. By multiplying both sides of this inequality by a_e and summing the resulting inequalities over all edges $e \in E$ we find that

$$\sum_{s\in S} \sum_{T\in\mathcal{T}_s} L_{\mathbf{a}}(T) r_T \le \sum_{e\in E} a_e C_e. \tag{2}$$

A lower bound for the left-hand side (LHS) of the preceding inequality is obtained when all sessions are routed along their shortest spanning trees: $\sum_{s\in S} \ell_{\mathbf{a}}(s)R_s \le \sum_{s\in S}\sum_{T\in\mathcal{T}_s} L_{\mathbf{a}}(T)r_T \le \sum_{e\in E} a_e C_e$. As we assume that the rate vector is on the hyperplane given by $\sum_{s\in S} \ell_{\mathbf{a}}(s)R_s = \sum_{e\in E} a_e C_e$, it follows that Condition 1) holds. To arrive at a contradiction, suppose next that Condition 2) is invalid. Hence, the rate-tuple $R = (R_1, \ldots, R_{|S|})$ is also routing-feasible in network G with link capacities C'_e for $e \in E$ in which $C'_e \le C_e$ for all e with strict inequality for at least one value of e with $a_e > 0$. The extended Japanese theorem (1) implies that

$$\sum_{s\in S} \ell_{\mathbf{a}}(s)R_s \le \sum_{e\in E} a_e C'_e < \sum_{e\in E} a_e C_e, \tag{3}$$

which contradicts $\sum_{s\in S} \ell_{\mathbf{a}}(s)R_s = \sum_{e\in E} a_e C_e$. Thus Condition 2) holds.

To establish sufficiency, consider a routing-feasible rate vector which satisfies Conditions 1) and 2). The argument for constraint (2) applies for any routing-feasible point, and Condition 2) implies that (2) can be replaced by $\sum_{s\in S}\sum_{T\in\mathcal{T}_s} L_{\mathbf{a}}(T)r_T = \sum_{e\in E} a_e C_e$. By Condition 1) we know that

$$\sum_{s\in S} \ell_{\mathbf{a}}(s)R_s = \sum_{s\in S}\sum_{T\in\mathcal{T}_s} L_{\mathbf{a}}(T)r_T = \sum_{e\in E} a_e C_e. \tag{4}$$

Hence the rate vector R is on the hyperplane corresponding to $\mathbf{a}$, completing the proof. ■

As we will next see, the true significance of the vector of edge distances in the extended Japanese theorem lies in the collection of shortest routing paths and trees for that distance vector used by the various unicast and multicast sessions; this can be viewed as a variation of Wardrop's principle [29].

Proposition 2.5: Consider two distance vectors, $\mathbf{a} = (a_1, \ldots, a_{|E|})$ and $\mathbf{b} = (b_1, \ldots, b_{|E|})$. If

1) for every edge $e \in E$, $a_e = 0$ implies $b_e = 0$, and
2) for every session $s \in S$ and tree $T \in \mathcal{T}_s$, $L_{\mathbf{a}}(T) = \ell_{\mathbf{a}}(s)$ implies $L_{\mathbf{b}}(T) = \ell_{\mathbf{b}}(s)$,

then the inequality corresponding to distance vector $\mathbf{a}$ is redundant in defining polytope P given the inequality corresponding to distance vector $\mathbf{b}$.

Before we prove this result, we will discuss an example of it. Consider an undirected ring network with $V = \{1, 2, 3\}$ which supports all possible unicast and multicast sessions. We represent session s as $\nu_s \to D_s$. Then our set of sessions is given by $S = \{1\to 2,\ 2\to 1,\ 2\to 3,\ 3\to 2,\ 3\to 1,\ 1\to 3,\ 1\to\{2,3\},\ 2\to\{1,3\},\ 3\to\{1,2\}\}$. Let $e_1 = \{1,2\}$, $e_2 = \{2,3\}$, $e_3 = \{3,1\}$. Suppose $C_1 = C_2 = C_3 = 1$ and $\mathbf{a} = (2,1,3)$. It is straightforward to confirm that

$$P = \left\{ R \in \mathbb{R}^{|S|} : 0 \le R_i, \sum_{s\in S} \ell_{\mathbf{a}}(s)R_s \le \sum_{e\in E} a_e C_e \quad \text{for all nonnegative integral distance vectors } \mathbf{a} \right\}. \tag{1}$$

- $\ell_{\mathbf{a}}(1 \to 2) = \ell_{\mathbf{a}}(2 \to 1) = 2$ and the shortest path is e_1,
- $\ell_{\mathbf{a}}(2 \to 3) = \ell_{\mathbf{a}}(3 \to 2) = 1$ and the shortest path is e_2,
- $\ell_{\mathbf{a}}(3 \to 1) = \ell_{\mathbf{a}}(1 \to 3) = 3$ and both paths are shortest, and
- $\ell_{\mathbf{a}}(1 \to \{2,3\}) = \ell_{\mathbf{a}}(2 \to \{1,3\}) = \ell_{\mathbf{a}}(3 \to \{1,2\}) = 3$ and the shortest tree is $\{e_1, e_2\}$.

Therefore, the halfspace resulting from distance vector $\mathbf{a}$ is shown in (5) at the bottom of the page. Next take $\mathbf{b} = (1,0,1)$. Observe that the shortest paths and shortest trees for each session under distance vector $\mathbf{a}$ continue to be shortest paths and shortest trees for the sessions under $\mathbf{b}$, although $\mathbf{b}$ has a second shortest path for unicast sessions $1 \to 2$ and $2 \to 1$ and a second shortest tree for the multicast sessions. The halfspace resulting from $\mathbf{b}$ is

$$\begin{aligned}(R_{1 \to 2} + R_{2 \to 1}) &+ (R_{3 \to 1} + R_{1 \to 3}) \\ &+ \left(R_{1 \to \{2,3\}} + R_{2 \to \{1,3\}} + R_{3 \to \{1,2\}}\right) \\ &\leq C_1 + C_3 = 2. \end{aligned} \tag{6}$$

The proposition stipulates that (5) is redundant for specifying the routing capacity region given (6). The proof establishes that every routing-feasible rate-tuple like $R_{1 \to 2} = R_{2 \to 3} = R_{3 \to 1} = 1$, $R_s = 0$, $s \notin \{1 \to 2,\ 2 \to 3,\ 3 \to 1\}$ which satisfies (5) with equality necessarily satisfies (6) with equality. The rate-tuple $R_{1 \to \{2,3\}} = 2$, $R_s = 0$, $s \notin \{1 \to \{2,3\}\}$ exemplifies a *routing infeasible* rate-tuple which satisfies (5) with equality; notice that four units of capacity are needed to route two units of multicast traffic, while the network has only three units of capacity.

Proof: Consider a routing-feasible rate vector R on the hyperplane defined by $\mathbf{a}$. By Lemma 2.4, Condition 1), every session is routed only along the shortest paths and trees for $\mathbf{a}$, and hence by assumption only along the shortest paths and trees for $\mathbf{b}$. Furthermore, note that any edge e with $b_e > 0$ must have $a_e > 0$ by assumption, and hence this edge must be fully utilized by Lemma 2.4, Condition 2). By Lemma 2.4, it follows that the routable point also is on the hyperplane defined for distance vector $\mathbf{b}$. Therefore, by Lemma 2.3, the bound corresponding to $\mathbf{a}$ is redundant given the inequality corresponding to $\mathbf{b}$. ■

We offer an alternate algebraic proof for Proposition 2.5 in Appendix III. Proposition 2.5 provides a powerful algorithmic technique for deriving the minimal set of inequalities for describing polytope P, and we next apply it to two communication problems on undirected ring networks.

III. The Routing Rate Region in Ring Networks

In this section we focus on the ring network $G(V,E)$, with set of vertices $V = \{1,\ldots,n\}$, and set of edges $E = \{1,\ldots,n\}$, as illustrated in Fig. 1. As an additional notation for rings, let $L_{\mathbf{a}}(p,q)$ denote the distance in the clockwise direction between vertices p and q assuming distance vector $\mathbf{a}$. As a first step in understanding the general multiple multicast problem on undirected ring networks, we focus here on the analysis of two special cases. In the first case, we consider a ring network in which each session $s \in S$ is a *line* session; i.e., the source ν_s and all destinations D_s form a sequence (in any order) of adjacent vertices in the network. In the second case, we study the ring network problem with multiple unicast and broadcast sessions. These special cases already require new techniques, and we are unaware of similar analyses in the literature. Our approach in each case is to show that for an arbitrary nontrivial distance vector $\mathbf{a} = (a_1,\ldots,a_n)$ we can construct a nontrivial distance vector $\mathbf{b} = (b_1,\ldots,b_n)$ with the following properties:

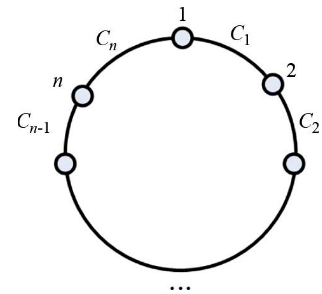

Fig. 1. An undirected ring network with n vertices.

- **Property 1:** $b_i = 0$ or 1 for all i,
- **Property 2:** $b_i = 0$ whenever $a_i = 0$,
- **Property 3:** for every session s, if $T \in \mathcal{T}_s$ and $\ell_{\mathbf{a}}(s) = L_{\mathbf{a}}(T)$, then $\ell_{\mathbf{b}}(s) = L_{\mathbf{b}}(T)$.

Hence, Proposition 2.5 implies that distance vector $\mathbf{a}$ can be eliminated by $\mathbf{b}$. It then follows that we can restrict our attention to distance vectors in the set $\{0,1\}^n$.

Remark 1: For ring networks it is sometimes more convenient to restate Property 3 in terms of the *complementary trees* of each session. Let $\mathcal{T}_s$ denote the set of routing trees of session s. Then the complementary tree $\bar{T}$ of a tree $T \in \mathcal{T}_s$ is the tree that remains after removing the edges and internal vertices of T from G. Let $\bar{\mathcal{T}}_s$ denote the set of all complementary trees corresponding to session s. For distance vector $\mathbf{a}$ let $c_{\mathbf{a}}$ denote the sum of all edge distances in the network. Given $\mathbf{a}$ and tree T, it follows that $L_{\mathbf{a}}(T) = c_{\mathbf{a}} - L_{\mathbf{a}}(\bar{T})$. Therefore, $\ell_{\mathbf{a}}(s) = \min_{T \in \mathcal{T}_s} L_{\mathbf{a}}(T) = c_{\mathbf{a}} - \max_{\hat{T} \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(\hat{T})$. We conclude that tree $T \in \mathcal{T}_s$ satisfies $\ell_{\mathbf{a}}(s) = L_{\mathbf{a}}(T)$ if and only if $L_{\mathbf{a}}(\bar{T}) = \max_{\hat{T} \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(\hat{T})$. Therefore, Property 3 is equivalent to the following.

- **Property 3':** for every session s, if $T \in \bar{\mathcal{T}}_s$ and $L_{\mathbf{a}}(T) = \max_{\hat{T} \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(\hat{T})$, then $L_{\mathbf{b}}(T) = \max_{\hat{T} \in \bar{\mathcal{T}}_s} L_{\mathbf{b}}(\hat{T})$.

Finally consider session s with $\nu_s = o$ and $D_s = \{d_1,\ldots,d_K\}$, where $d_1 < \cdots < d_i < o < d_{i+1} < \cdots < d_K$. Let us denote the path from vertex j to vertex k in the clockwise direction on

$$\begin{aligned}2(R_{1 \to 2} + R_{2 \to 1}) &+ (R_{2 \to 3} + R_{3 \to 2}) + 3(R_{3 \to 1} + R_{1 \to 3}) \\ &+ 3(R_{1 \to \{2,3\}} + R_{2 \to \{1,3\}} + R_{3 \to \{1,2\}}) \leq 2C_1 + C_2 + 3C_3 = 6. \end{aligned} \tag{5}$$

the ring G by $G(j,k)$. Then it is easy to verify that: See (6a) at the bottom of the page.

We next provide an algorithm for constructing $\mathbf{b} \in \{0,1\}^n$ for a given $\mathbf{a}$.

A. Algorithm for Constructing a Binary Distance Vector $\mathbf{b}$ for Distance Vector $\mathbf{a}$ for the Case of Line Sessions

Consider a set of line sessions S. Let $A = \max\{a_1,\ldots,a_n\} > 0$ and take $m_1 < \cdots < m_N$ to be the set of indices of all edges of maximum distance in $\mathbf{a}$, so that $a_{m_1} = a_{m_2} = \cdots = a_{m_N} = A$. Without loss of generality, we can assume that $m_1 = 1$ and so $a_1 = A > 0$. We abuse notation somewhat and write $L_{\mathbf{a}}(v_1, v_2) = \sum_{i=v_1}^{v_2-1} a_i$ as the length of the clockwise path from vertex v_1 to vertex v_2. The following algorithm shows that every distance vector can be reduced to a binary distance vector.

1) Set $b_j = 0$ for all $j \in \{1,\ldots,n\}$ with $a_j = 0$.
2) Set $i = 1$.
3) Complete the following steps:
   a) Set $b_i = 1$.
   b) Search for an index j such that $L_{\mathbf{a}}(i+1, j) < A$, but $L_{\mathbf{a}}(i+1, j+1) \geq A$, and $j \leq n$. If such a j exists, it must be unique. In this case let $i = j$ and return to Step 3. If no such j exists, go on to Step 4).
4) Set each remaining edge distance in $\mathbf{b}$ to 0.

We illustrate the algorithm above with an example:

Example 1: Consider distance vector $\mathbf{a} = (3,1,3,0,1,2)$. Then $A = 3$ and we set $b_4 = 0$. We initialize $i = 1$ and set $b_1 = 1$. Since $L_{\mathbf{a}}(2,3) < A$ and $L_{\mathbf{a}}(2,4) \geq A$, we next set $i = 3$ and $b_3 = 1$. Because $L_{\mathbf{a}}(4,6) < A$ and $L_{\mathbf{a}}(4,1) \geq A$ we set $i = 6$, $b_6 = 1$. As we can not further increase i we next set $b_2 = b_5 = 0$. The output of the algorithm will be $\mathbf{b} = (1,0,1,0,0,1)$.

B. Proof of the Algorithm Performance

It is clear that the algorithm satisfies Properties 1 and 2; we next show that it also satisfies Property 3. Fix a line session $s \in S$. For convenience, we relabel the vertices of the ring so that the source and all destinations for the session s are all on a line starting at vertex 1 and ending at vertex $|D_s|+1$. (Note that we may now have $m_1 \neq 1$.)

Consider the set of complementary trees for session s

$$\bar{\mathcal{T}}_s = \{G(1,2), G(2,3), \ldots, G(|D_s|, |D_s|+1), G(|D_s|+1, 1)\}.$$

Observe that

$$\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(T) = \max\{a_1, a_2, \ldots, a_{|D_s|}, L_{\mathbf{a}}(|D_s|+1, 1)\}.$$

To show that Property 3' holds we must verify that if $T \in \bar{\mathcal{T}}_s$ and $L_{\mathbf{a}}(T) = \max_{\hat{T} \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(\hat{T})$, then $L_{\mathbf{b}}(T) = \max_{\hat{T} \in \bar{\mathcal{T}}_s} L_{\mathbf{b}}(\hat{T})$. We begin with some lemmas.

Lemma 3.1: The vector $\mathbf{b}$ produced by the algorithm satisfies $b_{m_1} = b_{m_2} = \cdots = b_{m_N} = 1$.

Proof: By Step 3) of the algorithm we have $b_{m_1} = 1$. To arrive at a contradiction, for some $i > m_1$ suppose $b_i = 1$, j is the next smallest integer for which $b_j = 1$, and there is some $k > 1$ with $i < m_k < j$ and $b_{m_k} = 0$. Then Step 3) of the algorithm implies that $L_{\mathbf{a}}(i+1, j) < A$. However, since $m_k < j$, it follows that $L_{\mathbf{a}}(i+1, j) \geq L_{\mathbf{a}}(i+1, m_k+1) \geq a_{m_k} = A$, which is a contradiction. Therefore $b_{m_k} = 1$. ■

Lemma 3.2: If $L_{\mathbf{a}}(i,j) < A$, then there is at most one edge $k \in \{i,\ldots,j-1\}$ for which $b_k = 1$.

Proof: Suppose instead that there are $k, l \in \{i,\ldots,j-1\}$ with $k < l$, $b_k = b_l = 1$, and $b_{k+1} = \cdots = b_{l-1} = 0$. Then Step 3b) implies that $L_{\mathbf{a}}(k+1, l) < A$ and $L_{\mathbf{a}}(k+1, l+1) \geq A$. However, it then follows that $L_{\mathbf{a}}(i,j) \geq L_{\mathbf{a}}(k+1, l+1) \geq A$, which contradicts the assumption that $L_{\mathbf{a}}(i,j) < A$. ■

Lemma 3.3: If $L_{\mathbf{a}}(i,j) = A$, then there is exactly one edge $k \in \{i,\ldots,j-1\}$ for which $b_k = 1$.

Proof: Suppose first that there is no edge $k \in \{i,\ldots,j-1\}$ with $b_k = 1$. Let $r < i$ be the largest integer for which $b_r = 1$. Then $L_{\mathbf{a}}(r+1, j) < A$. However, since $A = L_{\mathbf{a}}(i,j) \leq L_{\mathbf{a}}(r+1, j)$ this can not happen. Next suppose that there are k and $l \in \{i,\ldots,j-1\}$ with $b_k = b_l = 1$ and $b_{k+1} = \cdots = b_{l-1} = 0$. By Step 1), $b_k = 1$ implies $a_k > 0$. By Step 3b), since $b_{k+1} = \cdots = b_{l-1} = 0$ and $b_l = 1$, it follows that $L_{\mathbf{a}}(k+1, l+1) \geq A$. Observe that $A = L_{\mathbf{a}}(i,j) \geq a_k + L_{\mathbf{a}}(k+1, l+1) > A$, which is a contradiction. ■

Lemma 3.4: If $L_{\mathbf{a}}(i,j) > A$, then there exists $k \in \{i,\ldots,j-1\}$ such that $b_k = 1$.

Proof: Suppose that there is no such k with $b_k = 1$, and let r be the largest integer less than i with $b_r = 1$. Then $L_{\mathbf{a}}(r+1, j) < A$. This contradicts the fact that $A < L_{\mathbf{a}}(i,j) \leq L_{\mathbf{a}}(r+1, j)$. ■

Since $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(T) \geq \max\{a_1,\ldots,a_n\}$, there are two possibilities for $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(T)$.

1) $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(T) = A$
2) $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(T) > A$

We consider the following three cases; the first two correspond to $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(T) = A$, and the third corresponds to $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(T) > A$. In each case we find the maximum length complementary trees with respect to $\mathbf{a}$ and show that they are also maximum length with respect to $\mathbf{b}$.

- First suppose that $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(T) = A$ and $L_{\mathbf{a}}(|D_s|+1, 1) < A$. Then by Lemma 3.1 we have $b_{m_1} = b_{m_2} = \cdots = b_{m_N} = 1$, and by Lemma 3.2 at most one among the edges in

$$\bar{\mathcal{T}}_s = \{G(d_1, d_2), \ldots, G(d_i, o), G(o, d_{i+1}), \ldots, G(d_{K-1}, d_K), G(d_K, d_1)\}. \tag{6a}$$

$\{|D_s|+1, |D_s|+2, \ldots, n\}$ will have a unit edge distance in $\mathbf{b}$. Therefore, $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{b}}(T) = 1$. Also, if for $T \in \bar{\mathcal{T}}_s$, $L_{\mathbf{a}}(T) = \max_{\hat{T} \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(\hat{T}) = A$, then $T = G(m_i, m_i+1)$ for some $m_i \in \{1, \ldots, |D_s|\}$. Since $b_{m_i} = \max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{b}}(T) = 1$, it follows that Property 3' is satisfied.

- Next suppose that $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(T) = A$ and $L_{\mathbf{a}}(|D_s|+1, 1) = A$. Then by Lemma 3.1 we have $b_{m_1} = b_{m_2} = \cdots = b_{m_N} = 1$, and by Lemma 3.3, exactly one among the edges in $\{|D_s|+1, |D_s|+2, \ldots, n\}$ should have a unit edge distance in $\mathbf{b}$. Hence $L_{\mathbf{b}}(|D_s|+1, 1) = 1$ and $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{b}}(T) = 1$. Also, if for $T \in \bar{\mathcal{T}}_s$, $L_{\mathbf{a}}(T) = \max_{\hat{T} \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(\hat{T}) = A$, then either $T = G(m_i, m_i+1)$ for some $m_i \in \{1, \ldots, |D_s|\}$ or $T = G(|D_s|+1, n)$, and since $b_{m_i} = L_{\mathbf{b}}(|D_s|+1, 1) = \max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{b}}(T) = 1$, it follows that Property 3' is satisfied.
- Finally, suppose $L_{\mathbf{a}}(|D_s|+1, 1) > A$. Then by Lemma 3.4, at least one among the edges in $\{|D_s|+1, |D_s|+2, \ldots, n\}$ should have a unit edge distance in $\mathbf{b}$, and hence $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{b}}(T) = L_{\mathbf{b}}(|D_s|+1, 1)$. Also, if for $T \in \bar{\mathcal{T}}_s$, $L_{\mathbf{a}}(T) = \max_{\hat{T} \in \bar{\mathcal{T}}_s} L_{\mathbf{a}}(\hat{T})$, then $T = G(|D_s|+1, n)$. Since $\max_{T \in \bar{\mathcal{T}}_s} L_{\mathbf{b}}(T) = L_{\mathbf{b}}(|D_s|+1, 1)$, we have that Property 3' is satisfied.

We have now shown that for any line session, our algorithm generates a binary distance vector $\mathbf{b}$ that satisfies Properties 1, 2, and 3. Thus, each distance vector with a distance greater than one can be reduced to a binary distance vector. Therefore, by Proposition 2.5 the routing capacity region can be determined by all binary distance vectors.

C. Algorithm for Constructing a Binary Distance Vector $\mathbf{b}$ *for Distance Vector* $\mathbf{a}$ *for the Case of Unicast and Broadcast Sessions*

We next consider the routing capacity region of a ring with a set of sessions S such that $|D_s| = 1$ or $|D_s| = n-1$ for all $s \in S$. We again prove that binary distance vectors suffice for describing polytope P by constructing a binary vector $\mathbf{b}$ which eliminates a given distance vector $\mathbf{a}$.

We first assume all edge distances in $\mathbf{a}$ are positive and subsequently extend the algorithm to general distance vectors with some zero elements. The heart of the algorithm is the following *Basic Generation Procedure:*

1) If all a_i are equal, then set $b_i = 1$ for $i \in \{1, \ldots, n\}$. Otherwise, proceed to the next step.
2) Draw a circle $\mathcal{C}$, with points on its perimeter corresponding to each vertex of the ring so that the length of the arc between two adjacent points on the circle is proportional to the corresponding edge distance in $\mathbf{a}$.
3) From each point on the perimeter of $\mathcal{C}$ draw a diameter originating from that point.
4) If the arc corresponding to an edge on $\mathcal{C}$ intersects at least one diameter, then set the corresponding edge distance in $\mathbf{b}$ to one; otherwise set it to zero.

Example 2: Consider a ring network with 6 vertices and a distance vector $\mathbf{a} = (1,\ 2,\ 4,\ 2,\ 2,\ 3)$. We wish to find the corresponding binary distance vector $\mathbf{b} = (b_1, \ldots, b_6)$ according to the Basic Generation Procedure. We first draw the circle $\mathcal{C}$ and all diameters for $\mathbf{a}$ according to Steps 2) and 3) (see Fig. 2).

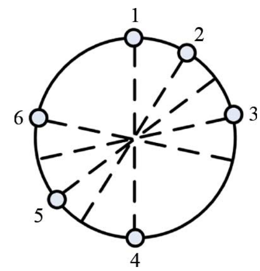

Fig. 2. An instance of applying the *Basic Generation Procedure* to a ring network in which $\mathbf{a} = (1,\ 2,\ 4,\ 2,\ 2,\ 3)$.

Since edges 2, 3, 4, and 5 are intersected by at least one diameter we set $b_2 = b_3 = b_4 = b_5 = 1$ and $b_1 = b_6 = 0$. The resulting binary distance vector is $\mathbf{b} = (0,\ 1,\ 1,\ 1,\ 1,\ 0)$.

The Basic Generation Procedure sometimes needs a correction to result in a $\mathbf{b}$ with the desired Properties; this depends on the path lengths of the different unicast sessions. We will see the appropriate method of constructing $\mathbf{b}$ for different cases and a proof of validity for each case:

First we categorize positive distance vectors $\mathbf{a}$ based on the path lengths of the different pairs of vertices into three types :

- **Type 1:** There is no pair of vertices with equal clockwise and counterclockwise routing path lengths by distance vector $\mathbf{a}$.
- **Type 2:** There is exactly one pair of vertices with equal clockwise and counterclockwise routing path lengths by distance vector $\mathbf{a}$.
- **Type 3:** There are multiple pairs of vertices with equal clockwise and counterclockwise routing path lengths by distance vector $\mathbf{a}$.

Theorem 3.5: If positive distance vector $\mathbf{a}$ is of Type 1, then the Basic Generation Procedure generates a distance vector $\mathbf{b}$ that satisfies Properties 1, 2, and 3.

Proof: See Appendix IV. ■

Theorem 3.6: If positive distance vector $\mathbf{a}$ is of Type 2, then either

1) the Basic Generation Procedure
 or
2) the Basic Generation Procedure followed by the change of a particular edge distance from 0 to 1 in $\mathbf{b}$ results in a legitimate vector $\mathbf{b}$ that satisfies Properties 1,2, and 3.

Proof: See Appendix V. ■

Theorem 3.7: If positive distance vector $\mathbf{a}$ is of Type 3, then $\mathbf{a}$ can be decomposed into several subvectors of Type 2 such that a combination of the binary distance subvectors corresponding to the subvectors of $\mathbf{a}$ results in a distance vector $\mathbf{b}$ that satisfies Properties 1,2, and 3.

Proof: See Appendix VI. ■

To complete the algorithm, consider distance vectors $\mathbf{a}$ with at least one element being equal to zero. By Proposition 2.5 for all i we set $b_i = 0$ if $a_i = 0$. Form a shorter distance vector, $\mathbf{a}' = (a'_1, a'_2, \ldots, a'_{n'})$, which is $\mathbf{a}$ without its zero elements. Observe that the length of a path between two vertices by $\mathbf{a}$ in a specific direction is equal to the length of the path between a corresponding pair of vertices for $\mathbf{a}'$ in the same direction.

Thus, given a suitable binary distance vector $\mathbf{b}'$ for $\mathbf{a}'$, we can find $\mathbf{b}$ by appropriately inserting zero edge distances into $\mathbf{b}'$. Clearly $\mathbf{b}$ preserves the shortest paths and broadcast trees for $\mathbf{a}$, completing the algorithm.

D. Concluding Remarks on the Extended Japanese Theorem

The algorithms that we provided in this section reduce a given distance vector into a binary distance vector in linear time in the size of the ring. The advantage of the reduction is that it provides a simple and finite characterization of the routing capacity region as opposed to the infinite set of inequalities. The following example illustrates the routing capacity region of a 3 vertex ring network.

Example 3: Consider a ring network supporting the unicast and broadcast sessions $S = \{1 \rightarrow 2, 1 \rightarrow 3, 2 \rightarrow \{1,3\}\}$. The routing capacity region of this problem can be derived by considering all binary distance vectors of length 3 and their corresponding inequalities as follows:

- the distance vector $(1,\ 1,\ 0)$ results in the inequality $R_{1 \rightarrow 2} + R_{2 \rightarrow \{1,3\}} \le C_1 + C_2$,
- the distance vector $(1,\ 0,\ 1)$ results in the inequality $R_{1 \rightarrow 2} + R_{1 \rightarrow 3} + R_{2 \rightarrow \{1,3\}} \le C_1 + C_3$,
- the distance vector $(0,\ 1,\ 1)$ results in the inequality $R_{1 \rightarrow 3} + R_{2 \rightarrow \{1,3\}} \le C_2 + C_3$,
- the distance vector $(1,\ 1,\ 1)$ results in the inequality $R_{1 \rightarrow 2} + R_{1 \rightarrow 3} + 2R_{2 \rightarrow \{1,3\}} \le C_1 + C_2 + C_3$.

Observe that binary distance vectors with all zeroes or a single one result in trivial inequalities since the shortest trees have length zero.

To conclude this section we point out that the bounds corresponding to binary distance vectors are not in general sufficient to characterize the routing capacity region of undirected rings with multiple multicast sessions. For example, we have the following lemma.

Lemma 3.8: For an undirected ring G with $n \ge 5$ vertices supporting all multicast sessions, the distance vector $\mathbf{a} = (x, 1, \ldots, 1)$ for $2 \le x \le \lfloor \frac{n-2}{2} \rfloor$ cannot be reduced to any $\mathbf{b} = (b_1, \ldots, b_n)$ with $\max\{b_1, \ldots, b_n\} < x$.

Proof: To arrive at a contradiction, assume that we have found a $\mathbf{b}$ with $\max\{b_1, \ldots, b_n\} < x$ that satisfies the conditions of Proposition 2.5. Let s be an arbitrary multicast session with $\nu_s = o$ and $D_s = \{d_1, d_2, \ldots, d_K\}$, where $d_1 < \cdots < d_i < o < d_{i+1} < \cdots < d_K$. Note that the set of complementary trees for session s is: See (6b) at the bottom of the page. Thus to satisfy the conditions of Proposition 2.5, the longest trees in $\bar{\mathcal{T}}_s$ with respect to $\mathbf{a}$ should remain longest under $\mathbf{b}$.

Consider the multicast session from 1 to $\{2, x+2, x+3, \ldots, n\}$. Here among the complementary trees $G(1,2), G(2, x+2), \ldots, G(n,1)$ there are two longest trees $G(1,2)$ and $G(2, x+2)$ under $\mathbf{a}$, and hence they should remain longest under $\mathbf{b}$. Thus, $b_1 = \sum_{i=2}^{x+1} b_i$. Likewise consider the multicast sessions from 1 to $\{2, 3, x+3, \ldots, n\}$, from 1 to $\{2, 3, 4, x+4, \ldots, n\}$, …, from 1 to $\{2, 3, \ldots, n-x, n\}$, and from 1 to $\{2, 3, \ldots, n-x+1\}$ to obtain the constraints

$$\begin{aligned} b_1 &= b_2 + b_3 + \cdots + b_{x+1} \\ b_1 &= b_3 + b_4 + \cdots + b_{x+2} \\ &\vdots \\ b_1 &= b_{n-x+1} + b_{n-x+2} + \cdots + b_n. \end{aligned} \tag{7}$$

Therefore

$$b_2 = b_{x+2},\ b_3 = b_{x+3},\ \ldots,\ b_{n-x} = b_n. \tag{8}$$

Next consider the multicast sessions from 1 to $\{3, x+4, \ldots, n\}$, from 1 to $\{3, 4, x+5, \ldots, n\}$, …, from 1 to $\{3, 4, \ldots, n-x-1, n\}$, and from 1 to $\{3, 4, \ldots, n-x\}$. The constraints maintaining the longest complementary trees with respect to $\mathbf{a}$ results in the following set of equalities:

$$\begin{aligned} b_1 + b_2 &= b_3 + b_4 + \cdots + b_{x+3} \\ b_1 + b_2 &= b_4 + b_5 + \cdots + b_{x+4} \\ &\vdots \\ b_1 + b_2 &= b_{n-x} + b_{n-x+1} + \cdots + b_n. \end{aligned} \tag{9}$$

Hence

$$b_3 = b_{x+4},\ b_4 = b_{x+5},\ \ldots,\ b_{n-x-1} = b_n. \tag{10}$$

Since $n - x \ge x + 2$, (8) and (10) imply

$$b_2 = b_3 \cdots = b_n \doteq b \tag{11}$$

and

$$b_1 = xb. \tag{12}$$

Since distance vector $\mathbf{b} \neq \mathbf{0}$, it follows that $b \neq 0$. Hence $b_1 = xb$ should be an integer bounded below by x, which is a contradiction. ■

Although we were able above to characterize the exact routing capacity region for two special cases, it appears difficult to apply our tools to arbitrary collections of multiple multicast sessions and/or arbitrary networks. However, these ideas offer some insights that help to further characterize the minimal set of distance vectors that define the routing capacity region for general multiple multicast networks. Indeed, in a recent work [11] we provide upper and lower bounds to show that the maximum edge distance needed for multiple multicast sessions in an undirected network grows exponentially with the size of the largest cycle of the network. The lower bound was obtained by demonstrating that a particular distance vector can not be reduced to another distance vector with a smaller maximal element. For the upper bound, observe that distance vectors are characterized by their shortest trees for the various sessions.

$$\bar{\mathcal{T}}_s = \{G(d_1, d_2), G(d_2, d_3), \ldots, G(d_i, o), G(o, d_{i+1}), G(d_{i+1}, d_{i+2}), \ldots, G(d_{K-1}, d_K), G(d_K, d_1)\}. \tag{6b}$$

Therefore for a given (and feasible) set of shortest trees, one can solve an integer programming problem to determine a distance vector with the same set of minimal trees. By investigating the size and complexity of the integer program and applying theorems of integer linear programming (see [24, Ch. 10]), we establish that the integer program always has distance vector solutions with elements that are exponentially large in the size of the largest cycle of the network.

IV. Network Coding Bounds

The set of routing bounds corresponding to binary distance vectors provides an inner bound to the network coding capacity region. To show that these bounds are tight for our two problems, we next present an information theoretic argument to establish the same bounds as outer bounds to the network coding capacity region. We say that a session is of *Type 1* if it is a line session and is of *Type 2* if it is a unicast or broadcast session. For $i \in \{1,2\}$ we say the set S of sessions is of Type i when every $s \in S$ is of Type i. We first prove the following useful lemma:

Lemma 4.1: For a ring with n vertices supporting a set of sessions of Type 1 or of Type 2, every routing bound corresponding to a binary distance vector with m ones, $m \leq n$, is equivalent to a routing bound for a ring with m vertices, where the distance vector for this latter network is the all-ones vector.

Proof: For the ring with n vertices and a binary distance vector, create a possibly smaller ring by successively replacing the vertices u and v with one vertex if u and v have zero distance between them. The routing bound corresponding to the original binary distance vector is clearly the same as the routing bound for the new ring with an all-ones distance vector. ∎

The proof that this routing bound is a network coding bound is developed next. It is easy to verify that the sessions of Type 1 or 2 will still be of the same type for the smaller network. Therefore we hereafter only consider distance vectors of the form $\mathbf{b}_n = (1,\ldots,1)$ for a ring G with $n \geq 2$ vertices.

First consider a ring with $n = 2$; here there are only two sessions, s_1 from 1 to 2 and s_2 from 2 to 1. The routing bound for this case, i.e., for $\mathbf{b}_2$ is $R_{s_1} + R_{s_2} \leq C_1 + C_2$, and this can easily be derived as a network coding bound using cut set bounds on edges 1 and 2.

Next consider a ring with $n = 3$ and distance vector $\mathbf{b}_3$. Here all multicast sessions are always both of Type 1 and of Type 2. The routing bound for this case is as follows:

$$\sum_{\{s:|D_s|=1\}} R_s + \sum_{\{s:|D_s|=2\}} 2R_s \leq C_1 + C_2 + C_3. \tag{13}$$

To show that (13) holds for network coding, we use the bidirected cut set bounds from [13] . Decompose each of the undirected capacities C_2, C_2, and C_3 into two unidirectional capacities: $C_1 = C_{12} + C_{21}$, $C_2 = C_{23} + C_{32}$, and $C_3 = C_{13} + C_{31}$, so that C_{pq} denotes the portion of the edge capacity which is directed from vertex p to vertex q. Then (13) can be obtained by summing the three bidirected cut set bounds derived from the pairs of directed edges $\big((1,2),(1,3)\big)$, $\big((2,1),(2,3)\big)$, and $\big((3,1),(3,2)\big)$.

For distance vector $\mathbf{b}_n$, $n \geq 4$, it turns out that the bidirected cut set bounds can not provide us with tight enough bounds for

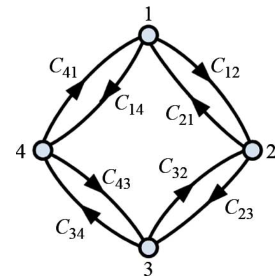

Fig. 3. A ring with four vertices.

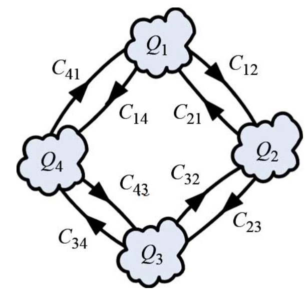

Fig. 4. A general ring with four sets of vertices.

the two types of sessions. In this case, we obtain our results for network coding via another set of bounds which are derived using the data processing inequality and the chain rule for mutual information.

Theorem 4.2: Consider the ring with four vertices illustrated in Fig. 3. Then for network coding:

$$\sum_{\nu_s \in \{1,3\}} R_s + \sum_{\nu_s=2,\ 4\in D_s} R_s + \sum_{\nu_s=4,\ 2\in D_s} R_s + \sum_{\{2,4\}\subseteq D_s} R_s \leq C_{12} + C_{32} + C_{14} + C_{34}. \tag{14}$$

Proof: See Appendix VII. ∎

For larger ring networks, a similar relationship can be established when vertices 1, 2, 3, and 4 are, respectively, replaced by four sets of neighboring vertices Q_1, Q_2, Q_3, and Q_4.

Proposition 4.3: For the ring in Fig. 4 we have

$$\sum_{s\in U_1} R_s + \sum_{s\in U_2} R_s + \sum_{s\in U_3} R_s + \sum_{s\in U_4} R_s \leq C_{12}+C_{32}+C_{14}+C_{34} \tag{15}$$

where

- $U_1 = \Big\{s : \nu_s \in Q_1, D_s \cap (Q_2 \cup Q_3 \cup Q_4) \neq \emptyset\Big\} \cup \Big\{\nu_s \in Q_3, D_s \cap (Q_1 \cup Q_2 \cup Q_4) \neq \emptyset\Big\}$
- $U_2 = \Big\{s : \nu_s \in Q_2, D_s \cap Q_4 \neq \emptyset\Big\}$
- $U_3 = \Big\{s : \nu_s \in Q_4, D_s \cap Q_2 \neq \emptyset\Big\}$
- $U_4 = \Big\{s : \nu_s \in (Q_1 \cup Q_3), D_s \cap Q_2 \neq \emptyset, D_s \cap Q_4 \neq \emptyset\Big\}$

and C_{ij} denotes the portion of the edge capacity directed from Q_i to Q_j.

Proof: Consider a network in which the number of vertices and the capacities C_{12}, C_{21}, C_{14}, C_{41}, C_{32}, C_{23}, C_{34}, and C_{43} are the same as in our original network in Fig. 3, but all other edge capacities are infinite. Assume the same traffic demands in the new network as in the original ring. A network coding solution that achieves the demands of the original ring is also a solution for this new ring. On the other hand, all capacities are infinite within any four groups of vertices Q_1, Q_2, Q_3, or Q_4, so each group can be treated as a single supervertex. Hence, our previous bound (14) for the ring of four vertices continues to hold for this modified ring. ■

We next use (15) to show that any network code must satisfy the routing bound corresponding to $\mathbf{b}_n$ in an undirected ring with $n \geq 4$ vertices. Consider (15), and let $E(i,j)$ denote the inequality derived by setting Q_2 and Q_4 to be two vertices i and j. For the values of i or j not between 1 and n, we consider their value modulo n in $E(i,j)$. We separately study the two cases of interest.

A. *Proof of the Network Coding Bound for Line Sessions for $n \geq 4$*

Consider a line session s. By using (15) we wish to find the coefficient of R_s in $E(i,j)$. First suppose that $|D_s \cap \{i,j\}| = 0$. By (15), s does not belong to U_2, U_3, or U_4. Furthermore since all source and destination vertices of s are adjacent, s is not in U_1. Therefore the coefficient of R_s in this case is zero. Next suppose that $|D_s \cap \{i,j\}| = 1$. In this case, s belongs to one of the sets U_1, U_2, or U_3, but not two of them together. Therefore the coefficient of R_s in this case is one. Finally suppose that $|D_s \cap \{i,j\}| = 2$. In this case s belongs to both U_1 and U_4, and therefore the coefficient of R_s is two. As a summary of these cases $E(i,j)$ can be written as follows:

$$\sum_{s \text{ is of Type } 1} |D_s \cap \{i,j\}| R_s \leq C_{(i-1)(i)} + C_{(i+1)(i)} + C_{(j-1)(j)} + C_{(j+1)(j)}. \quad (16)$$

Next we derive the network coding bound for this case and show that it is the same as the routing bound. Suppose that $n = 2m$ is an even integer. Then consider the sum $\sum_{i=1}^{m} E(i, m+i)$, which can be expanded as shown in (17) at the bottom of the page. Every directed capacity appears exactly once on the right-hand side (RHS) of (17), and thus it is equal to $\sum_{i=1}^{n} C_i$. Furthermore $\sum_{i=1}^{m} |D_s \cap \{i, i+m\}| = |D_s|$. Therefore (17) results in the following:

$$\sum_{s \text{ is of Type } 1} |D_s| R_s \leq \sum_{i=1}^{n} C_i. \quad (18)$$

Since for a line session s the source and destination vertices form a string of adjacent vertices, it follows that $\ell_{\mathbf{b}_n}(s)$ is the number of destination vertices of s on the ring. Hence, we obtain a network coding bound which is the same as the routing inequality for this case.

Next, suppose that $n = 2m+1$ is an odd number and consider the sum $0.5\left(\sum_{i=1}^{n} E(i, m+i)\right)$. Using (16), this sum can be expanded as (19)–(21) at the bottom of the page. Every directed capacity $C_{(i+1)(i)}$ appears in two terms of the summation of (21) which are the terms corresponding to $E(i, i+m)$ and $E(i-m, i)$. Therefore, (21) is $\sum_{i=1}^{n} C_i$. Next consider that in $\left(\sum_{i=1}^{n} |D_s \cap \{i, i+m\}|\right)$ in (20), every destination $d_s \in D_s$ is counted twice, namely in the terms corresponding to $E(d_s, d_s+m)$ and $E(d_s-m, d_s)$. Therefore, (20) is $0.5\left(\sum_{s \text{ is of Type } 1} (2|D_s|) R_s\right) = \sum_{s \text{ is of Type } 1} \ell_{\mathbf{b}_n}(s) R_s$, and, hence, the final result follows.

B. *Proof of the Network Coding Bound for Unicast and Broadcast Sessions for $n \geq 4$*

Consider again the cases $n = 2m$ and $n = 2m+1$ separately. Let $E_s(i,j)$ denote the coefficient of R_s on the LHS of $E(i,j)$ when s is of Type 2. First notice that by definition, for a unicast session s with source vertex ν_s and destination vertex d_s, if ν_s and d_s are on two sides of the ring which are separated by vertices i and j, or if d_s is i or j, then $E_s(i,j)$ is one; otherwise

$$\sum_{i=1}^{m} \sum_{s \text{ is of Type } 1} |D_s \cap \{i, i+m\}| R_s = \sum_{s \text{ is of Type } 1} \left(\sum_{i=1}^{m} |D_s \cap \{i, i+m\}| \right) R_s \leq \sum_{i=1}^{m} \left(C_{(i-1)(i)} + C_{(i+1)(i)} + C_{(i+m-1)(i+m)} + C_{(i+m+1)(i+m)} \right). \quad (17)$$

$$0.5 \left(\sum_{i=1}^{n} \sum_{s \text{ is of Type } 1} |D_s \cap \{i, i+m\}| R_s \right) \quad (19)$$

$$= 0.5 \left(\sum_{s \text{ is of Type } 1} \left(\sum_{i=1}^{n} |D_s \cap \{i, i+m\}| \right) R_s \right) \quad (20)$$

$$\leq 0.5 \left(\sum_{i=1}^{n} \left(C_{(i-1)(i)} + C_{(i+1)(i)} + C_{(i+m-1)(i+m)} + C_{(i+m+1)(i+m)} \right) \right). \quad (21)$$

it is zero. Furthermore, since a broadcast session s is a special case of a line session it follows that $E_s(i,j)$ is $|D_s \cap \{i,j\}|$.

For $n=2m$ we consider the sum $\sum_{i=1}^{m} E(i,m+i)$. Our previous arguments for line sessions implies that the RHS of this summation is $\sum_{i=1}^{n} C_i$. Equation (22), shown at the bottom of the page, is the LHS of summation. The coefficient of a broadcast session in (22) follows from the argument for line sessions. For a unicast session s consider $\sum_{i=1}^{m} E_s(i,i+m)$ and without loss of generality assume that $\nu_s=1$ and $d_s \le m$. Then the nonzero terms of the summation are $E_s(2,m+2)=\cdots=E_s(d_s,m+d_s)=1$. Since $\ell_{\mathbf{b}_n}(s)=d_s-1$, the coefficient of R_s will be $\ell_{\mathbf{b}_n}(s)$, and therefore the network coding inequality of the form

$$\sum_{s \text{ is of Type 2}} \ell_{\mathbf{b}_n}(s) R_s \le \sum_{i=1}^{n} C_i \tag{23}$$

is obtained in this setting, which is the same as the corresponding routing bound.

For $n=2m+1$, we can obtain the same inequality by instead considering $0.5\left(\sum_{i=1}^{n} E(i,m+i)\right)$. We consider the counterpart of (19)–(21) for this case. By the argument for line sessions, the RHS of this summation is $\sum_{i=1}^{n} C_i$. Next, we expand the LHS of this summation in (24), shown at the bottom of the page. For a unicast session s with $\nu_s=1$ and $d_s \le m+1$ consider $\sum_{i=1}^{n} E_s(i,i+m)$. The nonzero terms of this summation are $E_s(2,m+2)=\cdots=E_s(d_s,m+d_s)=1$ and $E_s(2,m+3)=\cdots=E_s(d_s,d_s+m+1)=1$. Since $\ell_{\mathbf{b}_n}(s)=d_s-1$, $\sum_{i=1}^{n} E_s(i,i+m)=2\ell_{\mathbf{b}_n}(s)$ and, therefore, the network coding inequality of the form

$$\sum_{s \text{ is of Type 2}} \ell_{\mathbf{b}_n}(s) R_s \le \sum_{i=1}^{n} C_i \tag{25}$$

follows for this setting and is the same as the corresponding routing inequality.

APPENDIX I
PROOF OF THE EXTENDED JAPANESE THEOREM

Here we prove Theorem 2.2. A rate vector $R=(R_1,\ldots,R_{|S|})$ is routable in a network $G(V,E)$ if and only if the following linear program has a solution for $\{r_T\}$:

1) $\sum_{T\in\mathcal{T}_s} r_T \ge R_s$ for every $s\in S$
2) $\sum_{s\in S}\sum_{T\in\mathcal{T}_s : e\in T} r_T \le C_e$ for every $e\in E$.
3) $0\le r_T$, for every $T\in\mathcal{T}_s$ and every $s\in S$.

Notice that the first set of inequalities can be changed to equalities, but it is equivalent and more convenient here to work with inequalities.

Label the elements of E and S from 1 to $|E|$ and from 1 to $|S|$, respectively. Also label the elements of $\mathcal{T}_s$ by $T_s^1,\ldots,T_s^{|\mathcal{T}_s|}$. For $e\in\{1,\ldots,|E|\}$, $s\in\{1,\ldots,|S|\}$, $T_s^j\in\{T_s^1,\ldots,T_s^{|\mathcal{T}_s|}\}$, let

$$\delta^s_{e,T_s^j}=\begin{cases}1, & e\in(T_s^j)\\ 0, & \text{otherwise.}\end{cases}$$

Define

$$\mathbf{r}=\left(r_{T_1^1},\ldots,r_{T_1^{|\mathcal{T}_1|}},\ldots,r_{T_{|S|}^1},\ldots,r_{T_{|S|}^{|\mathcal{T}_{|S|}|}}\right)^T \tag{26}$$

$$\mathbf{c}=\left(C_1,\ldots,C_{|E|},-R_1,\ldots,-R_{|S|},0,\ldots,0\right)^T \tag{27}$$

and matrix $\mathbf{M}$ as in (28) at the bottom of the next page. Then a routing-feasible assignment of $\{r_T : T\in\mathcal{T}_s, s\in S\}$ satisfies the following matrix inequality:

$$\mathbf{M}\mathbf{r}\le\mathbf{c}. \tag{29}$$

Farkas' lemma (see, e.g., [30, Sec. 1.4]) provides necessary and sufficient conditions for the feasibility of a system of linear inequalities. The following lemma applies Farkas' lemma to (29).

Lemma 4.4 (Farkas): There exists a solution to (29) if and only if every row vector $\mathbf{v}^T$ with $\mathbf{v}\ge\mathbf{0}$ and $\mathbf{v}^T\mathbf{M}=\mathbf{0}$ satisfies $\mathbf{v}^T\mathbf{c}\ge 0$.

$$\begin{aligned}\sum_{i=1}^{m}\sum_{s \text{ is a broadcast session}} |D_s\cap\{i,i+m\}|R_s+\sum_{i=1}^{m}\sum_{s \text{ is a unicast session}} E_s(i,i+m)R_s\\ =\sum_{s \text{ is a broadcast session}} \ell_{\mathbf{b}_n}(s)R_s+\sum_{s \text{ is a unicast session}}\left(\sum_{i=1}^{m}E_s(i,i+m)\right)R_s.\end{aligned} \tag{22}$$

$$\begin{aligned}0.5\left(\sum_{i=1}^{n}\sum_{s \text{ is a broadcast session}} |D_s\cap\{i,i+m\}|R_s\right)+0.5\left(\sum_{i=1}^{n}\sum_{s \text{ is a unicast session}} E_s(i,i+m)R_s\right)\\ =\sum_{s \text{ is a broadcast session}} \ell_{\mathbf{b}_n}(s)R_s+0.5\left(\sum_{s \text{ is a unicast session}}\left(\sum_{i=1}^{n}E_s(i,i+m)\right)R_s\right).\end{aligned} \tag{24}$$

We define v^T in (30) at the bottom of the page. Note that the steps of Fourier–Motzkin elimination maintain rational or integral rate coefficients throughout the procedure; this is why we need not consider irrational edge distances. The equation $\mathbf{v}^T\mathbf{M} = \mathbf{0}$ implies

$$\sum_{e\in E} v_e \delta^s_{e,T^j_s} - v_{|E|+s} - v_{|E|+|S|+z^j_s} = 0,\ s\in S, j\in\{1,\ldots,|\mathcal{T}_s|\} \quad (31)$$

where the expression for z^j_s appears in the last equation at the bottom of the page. Therefore by Lemma 4.4, for every $\mathbf{v}\geq\mathbf{0}$ satisfying (31), the inequality $\mathbf{v}^T\mathbf{c}\geq 0$ must hold. It can be written as

$$v_{|E|+1}R_1 + v_{|E|+2}R_2 + \cdots + v_{|E|+|S|}R_{|S|} \leq \sum_{e\in E} v_e C_e. \quad (32)$$

Fix a distance vector $\mathbf{a} = (a_1,\ldots,a_{|E|})$ and let $\mathbf{v_a} = \{\mathbf{v}\geq\mathbf{0} : (v_1,\ldots,v_{|E|}) = (a_1,\ldots,a_{|E|})\}$. Then for $\mathbf{v}\in\mathbf{v_a}$, (31) can be written as

$$L_{\mathbf{a}}(T^j_s) - v_{|E|+s} - v_{|E|+|S|+z^j_s} = 0, \quad s\in S, j\in\{1,\ldots,|\mathcal{T}_s|\} \quad (33)$$

and (32) can be written as

$$v_{|E|+1}R_1 + v_{|E|+2}R_2 + \cdots + v_{|E|+|S|}R_{|S|} \leq \sum_{e\in E} a_e C_e. \quad (34)$$

Since $v_{|E|+|S|+z^j_s}\geq 0$, then by (33), $v_{|E|+s}\leq L_{\mathbf{a}}(T^j_s)$ for every s and j. Therefore, for every $\mathbf{v}\in\mathbf{v_a}$, $v_{|E|+s}$ can be bounded from above by $\min_j L_{\mathbf{a}}(T^j_s) = \ell_{\mathbf{a}}(s)$. Observe that it is possible to choose $v_{|E|+s} = \ell_{\mathbf{a}}(s)$ for every s by setting $v_{|E|+s} = \ell_{\mathbf{a}}(s)$ and $v_{|E|+|S|+z^j_s} = L_{\mathbf{a}}(T^j_s) - \ell_{\mathbf{a}}(s)$. Next notice that the LHS of (34) is maximized among vectors in $\mathbf{v_a}$ when the values of $v_{|E|+s}$ are maximized; i.e., when $v_{|E|+s} = \ell_{\mathbf{a}}(s)$. Equivalently (34) holds if and only if the following inequality is satisfied:

$$\sum_{s\in S}\ell_{\mathbf{a}}(s)R_s \leq \sum_{e\in E} a_e C_e; \quad (35)$$

this is the routing bound corresponding to the distance vector $\mathbf{a}$.

Appendix II
Proof of Lemma 2.3

We begin by introducing some terminology and a result from [24, Ch. 8].

$$\mathbf{M} = \left(\begin{array}{ccc|ccc|ccc}
\delta^1_{1,T^1_1} & \cdots & \delta^1_{1,T_1^{|\mathcal{T}_1|}} & & & & \delta^{|S|}_{1,T^1_{|S|}} & \cdots & \delta^{|S|}_{1,T_{|S|}^{|\mathcal{T}_{|S|}|}} \\
\vdots & \ddots & \vdots & & \cdots & & \vdots & \ddots & \vdots \\
\delta^1_{|E|,T^1_1} & \cdots & \delta^1_{|E|,T_1^{|\mathcal{T}_1|}} & & & & \delta^{|S|}_{|E|,T^1_{|S|}} & \cdots & \delta^{|S|}_{|E|,T_{|S|}^{|\mathcal{T}_{|S|}|}} \\
\cline{1-3}\cline{7-9}
-1 & \ldots & -1 & & & & 0 & \ldots & 0 \\
0 & \ldots & 0 & & \cdots & & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & & & & 0 & \ldots & 0 \\
0 & \ldots & 0 & & & & -1 & \ldots & -1 \\
\cline{1-3}\cline{7-9}
-1 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
 & \ddots & & & & & & & \\
 & & -1 & & & & & & \\
 & & & -1 & & & & & \\
\vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
 & & & & & -1 & & & \\
 & & & & & & -1 & & \\
 & & & & & & & \ddots & \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & -1
\end{array}\right). \quad (28)$$

$$\mathbf{v}^T = \left(v_1,\ldots,v_{|E|},v_{|E|+1},\ldots,v_{|E|+|S|},v_{|E|+|S|+1},\ldots,v_{|E|+|S|+\sum_{s\in S}|\mathcal{T}_s|}\right). \quad (30)$$

$$z^j_s = \begin{cases} |\mathcal{T}_1| + |\mathcal{T}_2| + \cdots + |\mathcal{T}_{s-1}| + j, & s\in S, s>1, j\in\{1,\ldots,|\mathcal{T}_s|\} \\ j, & s=1, j\in\{1,\ldots,|\mathcal{T}_1|\}. \end{cases}$$

Let E be an arbitrary nonempty subset of the inequalities from (1) that define the polytope P of all routing-feasible rate vectors. Let F represent the collection of rate-vectors in P that satisfy each inequality in E with equality. If F is nonempty, it is called a *face* of the polytope P. A face F of P is said to be a *facet* of P if there is no face $F' \neq F$ of P for which $F \subset F'$. The following result from [24, Sec. 8.4], is central to the proof of Lemma 2.3.

Theorem 4.5 ([24, Section 8.4], Theorem 8.2): Suppose polytope P has no inequality which is always satisfied by equality. Further assume that $Ax \leq b$ is a minimal set of inequalities that define P. Let A_i^T denote the i^{th} row of A and let b_i denote the i^{th} element of column vector b. For each row i, there is a one-to-one correspondence between the defining halfspace $A_i^T x \leq b_i$ and a facet F_i of P given by $F_i = \{R \in P : A_i^T R = b_i\}$, where we represent rate-tuples as column vectors. Furthermore $Ax \leq b$ is the unique minimal representation of P up to the multiplication of inequalities by positive scalars.

To apply the preceding theorem we must first establish that there is no inequality in (1) that is always satisfied with equality. We assume that there is at least one Steiner tree in the network corresponding to each session $s \in S$ so that there is at least one rate-tuple with $R_s > 0$ for each $s \in S$. Next consider the inequality corresponding to the distance vector $\mathbf{a} \neq \mathbf{0}$. Let $F_{\mathbf{a}} = \{R \in P : \sum_{s\in S} \ell_{\mathbf{a}}(s) R_s = \sum_{e\in E} a_e C_e\}$. If R is a rate-tuple on the face $F_{\mathbf{a}}$, then R is not the all-zero vector. Furthermore, for any $0 < \epsilon < 1$, ϵR will be a feasible rate-tuple not on $F_{\mathbf{a}}$. Therefore, this inequality can not be always satisfied with equality.

To complete the proof of the lemma, observe that if for two distance vectors $\mathbf{a}$ and $\mathbf{b}$ the face $F_{\mathbf{a}}$ is included in $F_{\mathbf{b}}$, then $F_{\mathbf{a}}$ is not a facet of P. By the previous theorem, the inequality corresponding to distance vector $\mathbf{a}$ cannot be part of a minimal representation of P. Hence, distance vector $\mathbf{a}$ is redundant in the presence of distance vector $\mathbf{b}$.

APPENDIX III
AN ALGEBRAIC PROOF FOR PROPOSITION 2.5

Given distance vector $\mathbf{a} = (a_1, \ldots, a_{|E|})$, let M_s, $s \in \{1, \ldots, |S|\}$, denote the number of shortest routing trees for session s. Assume without loss of generality that for a fixed s, $L_{\mathbf{a}}(T_j^s)$ is nondecreasing with j so that

$$L_{\mathbf{a}}(T_1^s) = \cdots = L_{\mathbf{a}}(T_{M_s}^s) = \ell_{\mathbf{a}}(s).$$

Suppose that $(R_1, \ldots, R_{|S|})$ is a routable rate-tuple lying on the hyperplane $\sum_{s\in S} \ell_{\mathbf{a}}(s) R_s = \sum_{e\in E} a_e C_e$. By Condition 1) of Lemma 2.4 it follows that

$$\begin{aligned}\sum_{s\in S} \ell_{\mathbf{a}}(s) R_s &= \sum_{s\in S} \ell_{\mathbf{a}}(s) \sum_{j=1}^{M_s} r_{T_j^s} \\ &= \sum_{s\in S}\sum_{j=1}^{M_s} L_{\mathbf{a}}(T_j^s) r_{T_j^s} = \sum_{e\in E} a_e C_e.\end{aligned}$$

Define $\tilde{\delta}_{e,(s,j)}$ to be one if edge e is in the j^{th} shortest routing tree for session s with respect to distance vector $\mathbf{a}$ and 0 otherwise. Then by writing each $L_{\mathbf{a}}(T_j^s)$ as the sum of the edge distances a_e for which edge e occurs in the j^{th} shortest routing tree for session s, we see that

$$\begin{aligned}\ell_{\mathbf{a}}(s) &= L_{\mathbf{a}}(T_j^s) \\ &= \sum_{e\in E} \tilde{\delta}_{e,(s,j)} a_e, \quad s \in S, j \in \{1, \ldots, M_s\}. \quad (36)\end{aligned}$$

Condition 1) of Lemma 2.4 states that edge e is used by session s only if e is on a shortest routing tree for s. Therefore the partial flow of session s through routing tree T_j^s is $\tilde{\delta}_{e,(s,j)} r_{T_j^s}$, which is zero for $j > M_s$. Condition 2) of Lemma 2.4 states that every edge with a nonzero distance is fully utilized. Therefore if $a_e > 0$ for $e \in E$, then

$$\sum_{s\in S}\sum_{j=1}^{M_s} \tilde{\delta}_{e,(s,j)} r_{T_j^s} = C_e. \quad (37)$$

Set $\mathcal{S}$ to be the $|E| \times |E|$ diagonal matrix with diagonal entries $\sigma_e = \sum_{s\in S}\sum_{j=1}^{M_s} \tilde{\delta}_{e,(s,j)} r_{T_j^s} - C_e$, and set D to be the $|E| \times |E|$ diagonal matrix with diagonal entries $d_e = \frac{b_e}{a_e}$ when $a_e \neq 0$ and $d_e = 0$ when $a_e = 0$. It follows from (37) that $\mathcal{S}\mathbf{a}^T = \mathbf{0}$, and hence $D\mathcal{S}\mathbf{a}^T = \mathbf{0}$ as well. Diagonal matrices commute, so $\mathcal{S}D\mathbf{a}^T = \mathbf{0}$. Consider any edge $e \in E$. If $a_e \neq 0$ then $d_e a_e = b_e$. If $a_e = 0$, then $d_e a_e = 0$, and Condition 1) of Proposition 2.5 implies that $b_e = 0$. Therefore, $d_e a_e = b_e$ in this case as well. Thus $D\mathbf{a}^T = \mathbf{b}^T = (b_1, \ldots, b_{|E|})^T$ and it follows that $\mathcal{S}\mathbf{b}^T = \mathbf{0}$. Therefore

$$\begin{aligned}0 &= \sum_{e\in E} \sigma_e b_e \\ &= \sum_{s\in S}\sum_{j=1}^{M_s} \left(\sum_{e\in E} \tilde{\delta}_{e,(s,j)} b_e\right) r_{T_j^s} - \sum_{e\in E} b_e C_e. \quad (38)\end{aligned}$$

By Condition 2) of Proposition 2.5, $L_{\mathbf{b}}(T_1^s) = \cdots = L_{\mathbf{b}}(T_{M_s}^s) = \ell_{\mathbf{b}}(s)$ for each $s \in S$, and the counterpart to (36) is

$$\begin{aligned}\ell_{\mathbf{b}}(s) &= L_{\mathbf{b}}(T_j^s) \\ &= \sum_{e\in E} \tilde{\delta}_{e,(s,j)} b_e, \quad s \in S, j \in \{1, \ldots, M_s\}. \quad (39)\end{aligned}$$

Substituting (39) into (38) we obtain

$$\begin{aligned}0 &= \sum_{s\in S}\sum_{j=1}^{M_s} \ell_{\mathbf{b}}(s) r_{T_j^s} - \sum_{e\in E} b_e C_e \\ &= \sum_{s\in S} \ell_{\mathbf{b}}(s) R_s - \sum_{e\in E} b_e C_e\end{aligned}$$

and so $(R_1 \ldots, R_{|S|})$ does lie on the hyperplane $\sum_{s\in S} \ell_{\mathbf{b}}(s) R_s = \sum_{e\in E} b_e C_e$.

Thus, if a routable rate-tuple $(R_1, \ldots, R_{|S|})$ is on the hyperplane corresponding to $\mathbf{a}$, then it is also on the hyperplane corresponding to $\mathbf{b}$. Since the routing rate region can be described

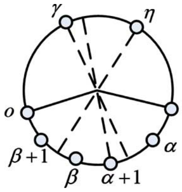

Fig. 5. An instance of a distance vector of Type 1 on a ring network.

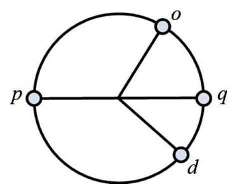

Fig. 6. An instance of Case 1) on a ring network.

in terms of its defining hyperplanes, the bound given by the hyperplane for $\mathbf{a}$ is redundant assuming we already have the bound given by the hyperplane given by $\mathbf{b}$.

Appendix IV
Proof of Theorem 3.5

Properties 1 and 2 are trivially satisfied in this case. Next we show that Property 3 also holds. We begin our discussion with unicast sessions. Consider two arbitrary vertices o and d of the network as in Fig. 5 and the unicast session s from o to d. Recall that for any two vertices i and j and distance vector $\mathbf{a}$, $L_{\mathbf{a}}(i,j)$ represents the length of the clockwise path from i to j on the ring with respect to the vector $\mathbf{a}$. Suppose $L_{\mathbf{a}}(d,o) < L_{\mathbf{a}}(o,d)$. We wish to show that $L_{\mathbf{b}}(d,o) \leq L_{\mathbf{b}}(o,d)$ so that the shortest path between two vertices remains shortest for $\mathbf{b}$. We first discuss the case for which there are at least two edges on the clockwise path from d to o with edge distance of 1 in $\mathbf{b}$. Consider an arbitrary edge α on the clockwise path from d to o such that $b_\alpha = 1$. Let β denote the next edge after α on the path from d to o in the clockwise direction with $b_\beta = 1$. Let $\mathcal{C}(o,d)$ denote the clockwise path from o to d. Since $b_\alpha = b_\beta = 1$, there must be at least two distinct vertices, say γ and η, on $\mathcal{C}(o,d)$ for which the diameter starting from these points will intersect the arcs corresponding to edges α and β (see Fig. 5). Next consider the diameter starting from point corresponding to vertex $\alpha+1$. This diameter should intersect $\mathcal{C}(o,d)$ at an edge between vertices η and γ. Therefore, for each pair of successive edges on the clockwise path from d to o with unit edge distances in $\mathbf{b}$ there is an edge on the clockwise path from o to d with unit edge distance in $\mathbf{b}$. Thus $L_{\mathbf{b}}(d,o) - 1 \leq L_{\mathbf{b}}(o,d)$. Furthermore, the two diameters starting from vertices o and d intersect $\mathcal{C}(o,d)$, and will produce two more unit edge distances in $\mathbf{b}$ that we have not yet counted. Thus we have $L_{\mathbf{b}}(d,o) \leq L_{\mathbf{b}}(o,d)$. To complete this argument, consider the case with $L_{\mathbf{b}}(d,o) = 0$; in this case apparently $0 = L_{\mathbf{b}}(d,o) \leq L_{\mathbf{b}}(o,d)$. Finally, if $L_{\mathbf{b}}(d,o) = 1$, we know that at least one of the diameters starting from o or d will produce a unit edge distance on $\mathcal{C}(o,d)$, and hence $1 = L_{\mathbf{b}}(d,o) \leq L_{\mathbf{b}}(o,d)$.

Next we show that Property 3 holds for broadcast sessions. Since the trees for routing broadcast sessions are the collection of paths consisting of all but one edge in the network, a shortest tree for a broadcast session corresponds to omitting an edge with maximal edge distance. Since $b_k \in \{0, 1\}$ for all k, we have to show that if a_i is a maximal edge distance in $\mathbf{a}$ then $b_i = 1$. To arrive at a contradiction, assume that $b_i = 0$. Then it follows that there is another edge j for which the diameters starting from vertices i and $i+1$ both intersect the arc corresponding to edge j. Hence $a_j > a_i$, which contradicts the maximality of a_i.

Appendix V
Proof of Theorem 3.6

To construct $\mathbf{b}$ for a Type 2 distance vector $\mathbf{a}$, we use the Basic Generation Procedure or the Basic Generation Procedure followed by the change of a particular edge distance from 0 to 1. Observe that Property 1 is trivially satisfied. Since we are only considering positive distance vectors $\mathbf{a}$, then Property 2 also holds. We next discuss Property 3. Assume that $\{p, q\}$ is the unique pair of vertices satisfying $L_{\mathbf{a}}(p,q) = L_{\mathbf{a}}(q,p)$. There are two subcases to consider:

1) Suppose the Basic Generation Procedure produces a vector $\mathbf{b}$ with $L_{\mathbf{b}}(p,q) = L_{\mathbf{b}}(q,p)$. Then in this case we do not need to make any changes to vector $\mathbf{b}$. Let us study a unicast session between two vertices o and d and show that Property 3 holds for it. If the clockwise path from p to q includes the clockwise path from o to d, then $L_{\mathbf{a}}(o,d) < L_{\mathbf{a}}(d,o)$. Similarly, $L_{\mathbf{b}}(o,d) \leq L_{\mathbf{b}}(p,q) = L_{\mathbf{b}}(q,p) \leq L_{\mathbf{b}}(d,o)$, which shows that Property 3 holds for the unicast session between o and d. A symmetric argument holds for the case where o and d are both located on the counterclockwise path from p to q.

   Next assume that o is located on the clockwise path from p to q and d is located on the counterclockwise path from p to q and that $L_{\mathbf{a}}(o,d) < L_{\mathbf{a}}(d,o)$ as depicted in Fig. 6. (By the assumption, $L_{\mathbf{a}}(o,d) \neq L_{\mathbf{a}}(d,o)$.) By the same argument as in the proof of Theorem 3.5, we can show that for each pair of successive edges on the clockwise path from o to q with unit distance in $\mathbf{b}$, there is an edge with unit distance on the clockwise path from d to p. Hence $L_{\mathbf{b}}(o,q) - 1 \leq L_{\mathbf{b}}(d,p)$. Observe that the diameter originating at vertex o will intersect an edge between d and p, and will produce another unit edge distance which we have not yet counted. (Note that vertex q can not produce any extra unit distance edge as the diameter originating at this vertex intersects the circle at p.) Thus, $L_{\mathbf{b}}(o,q) \leq L_{\mathbf{b}}(d,p)$. By symmetry, $L_{\mathbf{b}}(q,d) \leq L_{\mathbf{b}}(p,o)$. By summing these two inequalities we obtain

$$\begin{aligned} L_{\mathbf{b}}(o,d) &= L_{\mathbf{b}}(o,q) + L_{\mathbf{b}}(q,d) \\ &\leq L_{\mathbf{b}}(d,p) + L_{\mathbf{b}}(p,o) = L_{\mathbf{b}}(d,o) \end{aligned}$$

   which shows that Property 3 holds for the unicast session between o and d.

   The argument for broadcast sessions from Theorem 3.5 also holds for this case without any change, and so Property 3 is satisfied for broadcast sessions in this case as well.

2) Next suppose the lengths of the different paths between p and q do not remain the same for vector $\mathbf{b}$. We first describe how to modify distance vector $\mathbf{b}$ and we then show that the resulting distance vector $\mathbf{b}$ satisfies Property 3. Fig. 7 depicts the circle corresponding to the edge distances in $\mathbf{a}$. Observe that $a_p \neq a_q$ since the unicast session between

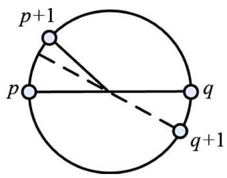

Fig. 7. An instance of Case 2) on a ring network.

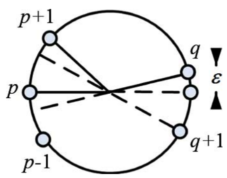

Fig. 8. The situation in Case 2) after moving q on $\mathcal{C}$.

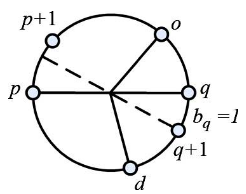

Fig. 9. The situation in Case 2) with the shorter path from o to d in the clockwise direction.

$p+1$ and $q+1$ do not have two equal length routing paths. Assume without loss of generality that $a_p > a_q$. Consider the clockwise path from $q+1$ to p and the clockwise path from p to q. With an argument similar to that for Type 1 distance vectors we can show that corresponding to every pair of successive edges with unit distance in $\mathbf{b}$ on the first path, there is an edge with unit distance on the second path. Thus we have $L_{\mathbf{b}}(q+1,p)-1 \leq L_{\mathbf{b}}(p,q)$. Observe that the diameter originating at vertex $q+1$ intersects the second path between vertices p and $p+1$ and produces another unit edge distance which we have not yet counted. Furthermore, edge q will not be intersected by any diameter and thus $b_q = 0$. We have the following relationship:

$$L_{\mathbf{b}}(q,p) = L_{\mathbf{b}}(q+1,p) \leq L_{\mathbf{b}}(p,q).$$

Now consider Fig. 8 in which we have moved the point corresponding to vertex q an arbitrarily small distance ϵ in the counterclockwise direction. If we apply the Basic Generation Procedure on this new set of edge distances and call the resulting binary set of distance vector $\mathbf{b}'$, then $b'_i = b_i$ for all edges except for edge q which is intersected by the diameter originating at vertex p, and possibly for edge $p-1$ which is now intersected by the diameter originating at vertex q; i.e., $b'_{p-1} = b'_q = 1$. If we use the argument for Type 1 distance vectors here, we find that for every pair of successive unit edge distances in $\mathbf{b}'$ for the edges on the clockwise path from p to q there is another edge with distance one on the clockwise path from q to p, and thus $L_{\mathbf{b}'}(p,q) - 1 \leq L_{\mathbf{b}'}(q,p)$. On the other hand, note that the two diameters originating at vertices p and q both produce two edges with unit distance in $\mathbf{b}'$ which we have not yet counted. Thus $L_{\mathbf{b}'}(p,q)+1 \leq L_{\mathbf{b}'}(q,p)$. Using the relationship of elements in $\mathbf{b}'$ and $\mathbf{b}$ we have $L_{\mathbf{b}'}(p,q) = L_{\mathbf{b}}(p,q)$ and $L_{\mathbf{b}'}(q,p) = L_{\mathbf{b}}(q,p)+1+(1-b_{p-1})$. Thus we have

$$L_{\mathbf{b}}(p,q) \leq L_{\mathbf{b}}(q,p) + (1 - b_{p-1}).$$

Recall that $L_{\mathbf{b}}(q,p) \leq L_{\mathbf{b}}(p,q)$. Therefore the condition $L_{\mathbf{b}}(q,p) \neq L_{\mathbf{b}}(p,q)$ implies $b_{p-1} = 0$, and in this case we have $L_{\mathbf{b}}(p,q) = L_{\mathbf{b}}(q,p) + 1$. In order to obtain $L_{\mathbf{b}}(q,p) = L_{\mathbf{b}}(p,q)$ we manually change the value of b_q from zero to one in $\mathbf{b}$. We next show that Property 3 holds for distance vector $\mathbf{b}$.

First we consider the unicast sessions. For all unicast sessions where both the source and destination are on the clockwise or counterclockwise path from p to q we use the argument from the preceding case. Next consider the unicast session between two vertices o and d, where o is located on the clockwise path from p to q and d is located on the counterclockwise path from p to q on the ring. First assume that the shortest path between o and d by $\mathbf{a}$ is the clockwise path from o to d (see Fig. 9) . Using the argument for Type 1 distance vectors, we conclude that for every pair of successive unit edge distances on the clockwise path from o to q there is another unit edge distance on the clockwise path from d to p, and for every pair of successive unit edge distances on the clockwise path from $q+1$ to d there is another edge with distance of one on the clockwise path from p to o. Since we have reset $b_q = 1$, we obtain $L_{\mathbf{b}}(o,q)-1 \leq L_{\mathbf{b}}(d,p)$ and $L_{\mathbf{b}}(q,d)-2 \leq L_{\mathbf{b}}(p,o)$. Thus $L_{\mathbf{b}}(o,d) - 3 \leq L_{\mathbf{b}}(d,o)$. Now if the diameters starting at $q+1$ and d generate two distinct unit edge distances in $\mathbf{b}$, these two along with the one generated by the diameter starting at o will add three more ones to $L_{\mathbf{b}}(d,o)$ and result in $L_{\mathbf{b}}(o,d) \leq L_{\mathbf{b}}(d,o)$.

However, it might happen that the diameters starting at $q+1$ and d both intersect edge p on $\mathcal{C}$. In this case we use the following argument. We have $L_{\mathbf{b}}(p,q) = L_{\mathbf{b}}(q,p)$ as the result of the modifications. Therefore, $1 + L_{\mathbf{b}}(p+1,q) = 1 + L_{\mathbf{b}}(q+1,p)$ and, hence, $L_{\mathbf{b}}(p+1,q) = L_{\mathbf{b}}(q+1,p)$. However $L_{\mathbf{b}}(q+1,p) = L_{\mathbf{b}}(d,p)$ because there is no diameter intersecting the clockwise path between $q+1$ and d. To arrive at a contradiction, suppose that there is at least one diameter intersecting the clockwise path between $q+1$ and d. By assumption the diameters starting at $q+1$ and d both intersect edge p on $\mathcal{C}$, and it follows that any diameter that intersects the clockwise path between $q+1$ and d must originate at a vertex between vertices p and $p+1$. However, there is no vertex between p and $p+1$, and so there is no diameter intersecting the clockwise path between $q+1$ and d. Therefore $L_{\mathbf{b}}(q+1,p) = L_{\mathbf{b}}(d,p)$. Thus, $L_{\mathbf{b}}(p+1,q) = L_{\mathbf{b}}(d,p)$. Since $L_{\mathbf{b}}(p+1,q)+1 = L_{\mathbf{b}}(p+1,d)$ and $L_{\mathbf{b}}(d,p)+1 = L_{\mathbf{b}}(d,p+1)$, we have $L_{\mathbf{b}}(p+1,d) = L_{\mathbf{b}}(d,p+1)$. Finally

$$L_{\mathbf{b}}(o,d) \leq L_{\mathbf{b}}(p+1,d) = L_{\mathbf{b}}(d,p+1) \leq L_{\mathbf{b}}(d,o).$$

Next suppose the shortest path between o and d is the counterclockwise path (see Fig. 10). Using the argument for

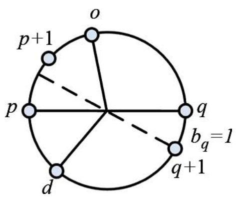

Fig. 10. The situation in Case 2) with the shorter path from o to d in the counterclockwise direction.

Type 1 distance vectors as in the previous instance, we conclude that $L_{\mathbf{b}}(d,p)-1 \leq L_{\mathbf{b}}(o,q)$ and $L_{\mathbf{b}}(p,o)-1 \leq L_{\mathbf{b}}(q,d)$. By accounting for the two unit edge distances produced by the diameters originating at o and d which we have not yet counted and by summing the two inequalities we get $L_{\mathbf{b}}(d,o) \leq L_{\mathbf{b}}(o,d)$. Hence, the shortest path for $\mathbf{a}$ will remain shortest for $\mathbf{b}$ and Property 3 holds for unicast sessions.

To complete our proof we need to show that Property 3' holds for broadcast sessions. Consider a broadcast session s and the set of its complementary trees. Since each routing tree for a broadcast session is the total ring after removing a single edge from it, then the set of complementary trees will be the set of all trees formed by single edges and their end vertices. Then to satisfy Property 3', we need to show that if edge e_0 satisfies $a_{e_0} = \max_{e\in E} a_e$, then $b_{e_0} = \max_{e \in E} b_e = 1$. First notice that as we saw in the argument for Type 1 distance vectors, the Basic Generation Procedure results in every edge with largest edge distance having a unit distance in $\mathbf{b}$. Moreover, the modifications for this case increase one edge distance from zero to one. Therefore all maximum length complementary trees with respect to $\mathbf{a}$ will have length 1 under $\mathbf{b}$, and hence Property 3' will be satisfied.

APPENDIX VI
PROOF OF THEOREM 3.7

Consider the case where there are exactly M pairs of vertices, say $\{p_1,q_1\},\{p_2,q_2\},\ldots,\{p_M,q_M\}$ for which $L_{\mathbf{a}}(p_i,q_i) = L_{\mathbf{a}}(q_i,p_i)$ for all $i \in \{1,\ldots,M\}$. Without loss of generality we assume that $1 = p_1 < p_2 < \cdots < p_M < q_1 < q_2 < \cdots < q_M \leq n$, as depicted in Fig. 11. To construct $\mathbf{b}$, we first decompose $\mathbf{a}$ into M subvectors $\mathbf{a}_1,\ldots,\mathbf{a}_M$, with

$$\mathbf{a}_i = (a_{p_i}, a_{p_i+1}, \ldots, a_{p_{i+1}-1}, a_{q_i}, a_{q_i+1}, \ldots, a_{q_{i+1}-1})$$

for $1 \leq i \leq M-1$ and $\mathbf{a}_M = (a_{p_M}, a_{p_M+1}, \ldots, a_{q_1-1}, a_{q_M}, a_{q_M+1}, \ldots, a_n)$.
From Fig. 11 it is easy to see that $L_{\mathbf{a}}(p_i,p_{i+1}) = L_{\mathbf{a}}(q_i,q_{i+1})$ for $1 \leq i \leq M-1$ and $L_{\mathbf{a}}(p_M,q_1) = L_{\mathbf{a}}(q_M,p_1)$. Therefore if we form the circle corresponding to subvector $\mathbf{a}_i$ and relabel the vertices on it from 1 to $p_{i+1}-p_i+q_{i+1}-q_i$, we obtain $L_{\mathbf{a}_i}(1,p_{i+1}-p_i+1) = L_{\mathbf{a}_i}(p_{i+1}-p_i+1,1)$ for $1 \leq i \leq M-1$ and $L_{\mathbf{a}_M}(1,q_1-p_M+1) = L_{\mathbf{a}_M}(q_1-p_M+1,1)$. Observe that each subvector $\mathbf{a}_i$, $1 \leq i \leq M$, in isolation corresponds to a circle with $|\mathbf{a}_i|$ vertices which has exactly one pair of vertices with equal length clockwise and counterclockwise paths between them, namely vertex 1 and vertex $p_{i+1}-p_i+1$; if there were more than one such pair, then there would be another pair p' and q' of vertices on the original ring such that $p_i < p' < p_{i+1}$, $q_i < q' < q_{i+1}$, and $L_{\mathbf{a}}(p',q') = L_{\mathbf{a}}(q',p')$, contradicting our initial assumption. Hence $\mathbf{a}_i$ is a Type 2 distance vector.

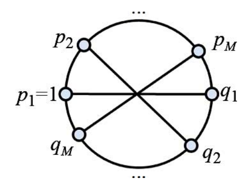

Fig. 11. The situation where there are several pairs of vertices with equal length clockwise and counterclockwise paths.

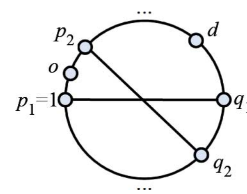

Fig. 12. The case where $d < q_1$.

For every $\mathbf{a}_i$ we use the argument for Type 2 distance vectors to construct a binary distance vector $\mathbf{b}_i$. We obtain $\mathbf{b} = (b_1,\ldots,b_n)$ from the relationships $(b_{p_i}, b_{p_i+1},\ldots,b_{p_{i+1}-1}, b_{q_i}, b_{q_i+1},\ldots,b_{q_{i+1}-1}) = \mathbf{b}_i$ for $1 \leq i \leq M-1$ and $(b_{p_M}, b_{p_M+1},\ldots,b_{q_1-1}, b_{q_M}, b_{q_M+1},\ldots,b_n) = \mathbf{b}_M$. Obviously $\mathbf{b}$ satisfies Properties 1 and 2. Next we prove that it also satisfies Property 3.

We first consider unicast sessions. The construction of $\mathbf{b}$ results in the following relationships:

$$\begin{aligned} L_{\mathbf{b}}(p_i,p_{i+1}) &= L_{\mathbf{b}}(q_i,q_{i+1}), \quad 1 \leq i \leq M-1, \\ L_{\mathbf{b}}(p_M,p_1) &= L_{\mathbf{b}}(q_M,q_1). \end{aligned} \tag{40}$$

Consider a pair o and d of vertices on the ring and the unicast session s between them. Assume without loss of generality that $p_1 \leq o \leq p_2$ and $L_{\mathbf{a}}(o,d) \leq L_{\mathbf{a}}(d,o)$, so that $o < d \leq q_2$. We must establish that $L_{\mathbf{b}}(o,d) \leq L_{\mathbf{b}}(d,o)$. To show this, first assume that $d < q_1$ (see Fig. 12) . In this case, it follows from (40) that

$$L_{\mathbf{b}}(o,d) \leq L_{\mathbf{b}}(p_1,q_1) = L_{\mathbf{b}}(q_1,p_1) \leq L_{\mathbf{b}}(d,o). \tag{41}$$

Next let $q_1 \leq d \leq q_2$ (see Fig. 13). Then by assumption [see (42) at the bottom of the next page]. It follows from (42) that

$$L_{\mathbf{a}}(o,p_2) + L_{\mathbf{a}}(q_1,d) \leq L_{\mathbf{a}}(d,q_2) + L_{\mathbf{a}}(p_1,o). \tag{43}$$

Since $\mathbf{b}_1$ satisfies Property 3 for $\mathbf{a}_1$ it follows from (43) and the definition of $\mathbf{b}$ that

$$L_{\mathbf{b}}(o,p_2) + L_{\mathbf{b}}(q_1,d) \leq L_{\mathbf{b}}(d,q_2) + L_{\mathbf{b}}(p_1,o). \tag{44}$$

Therefore (44) and (40) imply (45), shown at the bottom of the next page. Hence distance vector $\mathbf{b}$ satisfies Property 3 for the

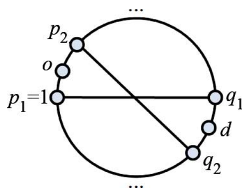

Fig. 13. The case where $q_1 \le d \le q_2$.

unicast sessions. To prove that this is also true for broadcast sessions, we have to show that all edges with largest edge distances for $\mathbf{a}$ have unit edge distances for $\mathbf{b}$. Let $p_i \le k \le p_{i+1}$ be an edge distance with largest distance for $\mathbf{a}$. Then it must correspond to an edge with largest distance for $\mathbf{a}_i$. Hence our earlier argument for Type 2 distance vectors establishes that the position corresponding to b_k in $\mathbf{b}_i$ is equal to one, and this proves our claim.

Appendix VII
Proof of the Network Coding Bound for a Ring With Four Vertices

Consider a ring network with four vertices. We replace every edge in the network with two oppositely directed edges and obtain the directed graph $G(V,E)$ of Fig. 3. Suppose that the network is clocked, i.e., a universal clock ticks N times. For this network we introduce the following notation for the network coding setting from time 1 to N.

- Let W_s denote the message of session s.
- Let $X_{ij}^{(t)}$ denote the bitstream of edge from i to j at time t and $X_{ij}^{k} = \left[X_{ij}^{(1)}, \ldots, X_{ij}^{(k)}\right]$.

Vertex i transmits the bitstream $X_{ij}^{(t)}$, $(i,j) \in E$ after clock tick $t-1$ and before clock tick t for $t = 1, \ldots, N$ and vertex j receives bitstream $X_{ij}^{(t)}$ at clock tick t. In a network coding solution, $X_{ij}^{(t)}$ is a function of $\{W_s : \nu_s = i\}$ and $\{X_{(i+1)(i)}^{t-1}, X_{(i-1)(i)}^{t-1}\}$. After time N, at every vertex i the received messages $\{W_s : i \in D_s\}$ with destination i can be decoded as a function of $\{W_s : \nu_s = i\}$ and $\{X_{(i+1)(i)}^{N}, X_{(i-1)(i)}^{N}\}$. For this setting we prove the following lemma.

Lemma 4.6: For any network and any network coding solution, there is a one to one correspondence between the set of messages in the network $\{W_s : s \in S\}$, and the set of bitstreams of the edges $\{X_{ij}^{N} : (i,j) \in E\}$.

Proof: Since the encoding functions at vertices are deterministic, a set of messages uniquely determine a set of bitstreams. Next suppose that there are two different realizations of $\{W_s : s \in S\}$, say $U = \{u_s : s \in S\}$ and $V = \{v_s : s \in S\}$ corresponding to a realization of bitstreams $\{X_{ij}^{N} : (i,j) \in E\}$, say $X = \{x_{ij}^{N} : (i,j) \in E\}$. It means that there is at least one session s_0 for which $u_{s_0} \neq v_{s_0}$. Next, we show that F which is another realization of messages and is a combination of messages in U and V as follows:

$$F = \{f_s : s \in S\}, \quad f_s = \begin{cases} u_s & \text{if } \nu_s \neq \nu_{s_0}, \\ v_s & \text{if } \nu_s = \nu_{s_0} \end{cases} \tag{46}$$

also results in the bitstream X. We use induction over time instances. For every vertex i, X_{ij}^{1} is a function of $\{W_s : \nu_s = i\}$. For $i \neq \nu_{s_0}$, the realization of X_{ij}^{1} corresponding to F is the same as its realization for U, and for $i = \nu_{s_0}$, the realization of X_{ij}^{1} corresponding to F is the same as its realization for V. Therefore, by assumption X_{ij}^{1} will have the same realization for U, V, and F. As the induction step, suppose that for time instance k, the realization of X_{ij}^{k} is the same for U, V, and F. Next, $X_{ij}^{(k+1)}$ is a function of $\{W_s : \nu_s = i\}$ and $\{X_{ji}^{k}, (j,i) \in E\}$. Therefore, the realization of $X_{ij}^{(k+1)}$ corresponding to F is equal to its realization corresponding to U if $i \neq \nu_{s_0}$ and to its realization corresponding to V if $i = \nu_{s_0}$. Thus, by the induction hypothesis, $X_{ij}^{(k+1)}$ will have the same realization for U, V, and F, which completes our induction.

Next consider a vertex $d \in D_{s_0}$. First notice that since $d \neq \nu_{s_0}$, the realization of $\{X_{jd}^{N}, (j,d) \in E\}$ and $\{W_s : \nu_s = d\}$ is the same for set of messages U and F. Therefore, d will decode the same messages for all sessions with destination d in both cases. But this contradicts the fact that W_{s_0} has two different realizations for U and F. Therefore, every set of bitstreams $\{X_{ij}^{N} : (i,j) \in E\}$ corresponds to a unique set of messages $\{W_s : s \in S\}$. ■

We apply the result of Lemma 4.6 to the ring network of Fig. 3. Furthermore in the ring network of Fig. 3 every realization of messages $\{W_s : \nu_s \in \{2,4\}\}$ and bitstreams $\{X_{ij}^{N} : i \in \{1,3\}, j \in \{2,4\}\}$ uniquely determines a realization of bitstreams $\{X_{ij}^{N} : i \in \{2,4\}, j \in \{1,3\}\}$. Therefore, together with Lemma 4.6, we conclude that every realization of $\{W_s : \nu_s \in \{2,4\}\}$ and $\{X_{ij}^{N} : i \in \{1,3\}, j \in \{2,4\}\}$ uniquely determines a realization of $\{W_s : s \in S\}$ and vice versa. Therefore, (47), shown at the top of the next page, holds. By expanding the RHS of (47), we obtain (48), shown at the top of the next page. Next we find a lower bound for $I(U_1; U_2)$. First notice that U_1 consists of the set of messages originating at vertex 2 and the bitstreams received by vertex 2. Therefore,

$$\begin{aligned} L_{\mathbf{a}}(o,d) &= L_{\mathbf{a}}(o,p_2) + L_{\mathbf{a}}(p_2,p_3) + \cdots + L_{\mathbf{a}}(p_M,q_1) + L_{\mathbf{a}}(q_1,d) \\ &\le L_{\mathbf{a}}(d,q_2) + L_{\mathbf{a}}(q_2,q_3) + \cdots + L_{\mathbf{a}}(q_M,p_1) + L_{\mathbf{a}}(p_1,o) \\ &= L_{\mathbf{a}}(d,o). \end{aligned} \tag{42}$$

$$\begin{aligned} L_{\mathbf{b}}(o,d) &= L_{\mathbf{b}}(o,p_2) + L_{\mathbf{b}}(p_2,p_3) + \cdots + L_{\mathbf{b}}(p_M,q_1) + L_{\mathbf{b}}(q_1,d) \\ &\le L_{\mathbf{b}}(d,q_2) + L_{\mathbf{b}}(q_2,q_3) + \cdots + L_{\mathbf{b}}(q_M,p_1) + L_{\mathbf{b}}(p_1,o) \\ &= L_{\mathbf{b}}(d,o). \end{aligned} \tag{45}$$

$$H\left(\{W_s : s \in S\}\right) = H\left(\{W_s : \nu_s \in \{2,4\}\}, \{X_{ij}^N : i \in \{1,3\}, j \in \{2,4\}\}\right). \tag{47}$$

$$\begin{aligned} H\left(\{W_s : s \in S\}\right) =& H\left(\{W_s : \nu_s = 2\}, X_{12}^N, X_{32}^N\right) + H\left(\{W_s : \nu_s = 4\}, X_{14}^N, X_{34}^N\right) \\ &- I\left(\underbrace{\{W_s : \nu_s = 2\}, X_{12}^N, X_{32}^N}_{U_1}; \underbrace{\{W_s : \nu_s = 4\}, X_{14}^N, X_{34}^N}_{U_2}\right). \end{aligned} \tag{48}$$

$$\begin{aligned} &H\left(\{W_s : s \in S\}\right) + H\left(\{W_s : \nu_s = 2,\ 4 \in D_s\}, \{W_s : \nu_s = 4,\ 2 \in D_s\}, \{W_s : \{2,4\} \subseteq D_s\}\right) \\ &\quad \le H\left(\{W_s : \nu_s = 2\}, X_{12}^N, X_{32}^N\right) + H\left(\{W_s : \nu_s = 4\}, X_{14}^N, X_{34}^N\right) \\ &\quad \le H\left(\{W_s : \nu_s = 2\}\right) + H\left(\{W_s : \nu_s = 4\}\right) + H\left(X_{12}^N\right) + H\left(X_{32}^N\right) + H\left(X_{14}^N\right) + H\left(X_{34}^N\right). \end{aligned} \tag{49}$$

$$\begin{aligned} &H\left(\{W_s : \nu_s \in \{1,3\}\}\right) + H\left(\{W_s : \nu_s = 2,\ 4 \in D_s\}, \{W_s : \nu_s = 4,\ 2 \in D_s\}, \{W_s : \{2,4\} \subseteq D_s\}\right) \\ &\quad \le H\left(X_{12}^N\right) + H\left(X_{32}^N\right) + H\left(X_{14}^N\right) + H\left(X_{34}^N\right). \end{aligned} \tag{50}$$

this set uniquely determines the messages destined for vertex 2. Hence, this set can be used to obtain the set of messages $M = \{W_s : \nu_s = 2,\ 4 \in D_s\} \cup \{W_s : \nu_s = 4,\ 2 \in D_s\} \cup \{W_s : \{2,4\} \subseteq D_s\}$. By an analogous argument for vertex 4 it follows that the set of messages M is a function of U_2. Thus the mutual information term in (48) can be written as $I(U_1, M; U_2, M)$. By expanding this term and using the data processing inequality we have

$$I(U_1, M; U_2, M) \ge I(M; M) = H(M).$$

We combine the preceding bound with (48) to establish (49), shown at the top of the page. By the independence of messages of different sessions and (49) we have (50), shown at the top of the page. Notice that $H\left(X_{ij}^N\right) \le NC_{ij}$ and $H\left(W_s\right) = NR_s$. It follows that for a ring with four vertices:

$$\begin{aligned} \sum_{s:\nu_s \in \{1,3\}} R_s + \sum_{s:\nu_s = 2,\ 4 \in D_s} R_s + \sum_{s:\nu_s = 4,\ 2 \in D_s} R_s \\ + \sum_{s:\{2,4\} \subseteq D_s} R_s \le C_{12} + C_{32} + C_{14} + C_{34}. \end{aligned} \tag{51}$$

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